A note on dual connections of Finsler spaces

by Tetsuya NAGANO*

Abstract

The author investigated the statistical structure for Finsler spaces in [N05]. In this paper, the dual connections of Finsler connections are studied.

Keywords: statistical structure, dual connection, Finsler connection, Finsler metric, vertical lift, horizontal lift, complete lift, sasaki lift, tangent bundle.

Introduction

In Information geometry, there are two key notions. They are notions of the statistical structure and the dual connection. The author investigated the statistical structure for Finsler spaces in [N05] and proposed the notion of the statistical structure for Finsler spaces on the 9th International Conference of Tensor Society at Sapporo in 2006.

In this paper, the author studies the dual connection of Finsler connections. The Finsler connection consists of a nonlinear connection and a linear connection on tangent bundle. The author investigates in detail when the linear connection consisting of the Finsler connection and the lifted metric of the Finsler metric satisfy the condition of the dual connection. The author studies the four cases, the vertical-, the horizontal-, the complete-, and the sasaki-lifts of the Finsler metric. From the obtained four results(Theorem 3.1 $\sim$ 3.4), we notice the condition that makes four results to be uniform(Corollary 3.1 $\sim$ 3.4). In addition, we can see that the symmetric property of the linear connection consisted of the Finsler connection, it was the obstacle for Finsler spaces, will be substituted to a more weekly condition(Remark 3.1).

In §1, the notion of the Finsler metric and the Finsler connection are stated. In §2, the four types of the lifted metric are stated. In §3, the dual connection of a linear connection consisted of the Finsler connection are investigated in detail in four cases.

In this paper, the author refers [A-N00], [M-I03] and obeys [M86] with respect to the notations of the covariant derivation and the position of its indices and refers [Y-I73] with respect to the typical lifts to its tangent bundles. In addition the author obeys [A-K05] with respect to the global notations of Finsler geometry.

1 Finsler metrics and Finsler Connections

Firstly, we state the notation of Finsler metrics and Finsler connections([A-K05],[M86]).

*Department of Info-media, Siebold University of Nagasaki. E-mail: hnagano@sun.ac.jp

Definition 1.1 *A function* $F : TM \to \mathbb{R}$ *is called a Finsler metric on* M *if*
1. $F(x,y) \geq 0$, *and* $F(x,y) = 0$ *if and only if* $y = 0$,
2. $F(x, \lambda y) = \lambda F(x,y)$ *for* $\forall \lambda \in \mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda > 0\}$,
3. $F(x,y)$ *is differentiable on* $TM^\times$
4. *the Hessian* (g_{ij}) *defined by*

$$g_{ij}(x,y) = \frac{\partial^2(\frac{1}{2}F^2)}{\partial y^i \partial y^j} \tag{1.1}$$

is regular.

Then the pair (M,F) *is called a Finsler space(or Finsler manifold). For each* $X \in T_xM$, *its norm* $||X||$ *is defined by* $||X|| = F(x,X)$.

A Finsler metric F is said to be *convex* if $F^2/2$ is *strictly convex* on each tangent space T_xM, that is, the Hessian (g_{ij}) is positive-definite. The convexity of F is equivalent to the one of the unit ball $B_x = \{y \in T_xM | F(x,y) \leq 1\}$. In addition, in this paper, we also call g_{ij} Finsler metric.

Next,

Definition 1.2 *Let* N *be a nonlinear connection and* ∇ *a linear connection on* TM, *respectively. If* ∇ *satisfies the following conditions*

$$\nabla_{\delta_j}\delta_i = F^r_{ij}\delta_r, \quad \nabla_{\delta_j}\partial_{\bar{i}} = F^r_{ij}\partial_{\bar{r}}, \quad \nabla_{\partial_{\bar{j}}}\delta_i = C^r_{ij}\delta_r, \quad \nabla_{\partial_{\bar{j}}}\partial_{\bar{i}} = C^r_{ij}\partial_{\bar{r}}, \tag{1.2}$$

then the pair $F\Gamma = (N, \nabla)$ *is called a Finsler connection on* M, *where* $(\delta_i, \partial_{\bar{i}}) = (\partial_i - N^r_i\partial_{\bar{r}}, \partial_{\bar{i}})$ *is the adapted frame of* TM *and* N^i_j *are coefficients of* N *on each local coordinate* (x^i, y^i).

We also call $(N^i_j, F^i_{jk}, C^i_{jk})$ the *coefficients of Finsler connection* $F\Gamma$ and denote by $F\Gamma = (N^i_j, F^i_{jk}, C^i_{jk})$.

Then we can see the following relations for the coframe $(dx^i, \delta y^i) = (dx^i, dy^i + N^i_j dx^j)$,

$$\nabla_{\delta_j}dx^i = -F^i_{rj}dx^r, \ \nabla_{\delta_j}\delta y^i = -F^i_{rj}\delta y^r, \ \nabla_{\partial_{\bar{j}}}dx^i = -C^i_{rj}dx^r, \ \nabla_{\partial_{\bar{j}}}\delta y^i = -C^i_{rj}\delta y^r. \tag{1.3}$$

2 Lifts of Finsler metrics

In general, the dual connection depends on the metric G and the connection ∇ (see §3), so we consider various semi-Riemannian metrics on $TM^\times$ derived from Finsler metric g_{ij}.

On each local coordinate (x^i, y^i)

(I) The case of Vertical lift

$$g^v = g_{ij}dx^i \otimes dx^j. \tag{2.1}$$

(II) The case of Horizontal lift

$$g^h = g_{ij}(\delta y^i \otimes dx^j + dx^i \otimes \delta y^j). \tag{2.2}$$

(III) The case of Complete lift

$$g^c = y^k g_{ij|k} dx^i \otimes dx^j + g_{ij}(\delta y^i \otimes dx^j + dx^i \otimes \delta y^j). \tag{2.3}$$

(IV) The case of Sasaki lift

$$g^s = g_{ij} dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j. \tag{2.4}$$

In this paper, we study the above four lifted metrics.

3 Dual connections

Firstly, we state the definition of the dual connection.

Definition 3.1 *Let G be a semi-Riemannian metric and ∇ a linear connection. If a linear connection ∇^* satisfies the following relation*

$$ZG(X,Y) = G(\nabla_Z X, Y) + G(X, \nabla^*_Z Y), \tag{3.1}$$

then ∇^* is called the dual connection of ∇ with respect to G, where X, Y, Z are vector fields.

Now, we set the following situation.
A Finsler space (M, F) and a Finsler connection $F\Gamma = (N, \nabla) = (N^i_j, F^i_{jk}, C^i_{jk})$ are given. Further another Finsler connection $F\Gamma^* = (N^*, \nabla^*) = (N^{*i}_{\ j}, F^{*i}_{\ jk}, C^{*i}_{\ jk})$ are given. In this case we have two frames $(\delta_i, \partial_{\bar{i}}) = (\partial_i - N^r_i \partial_{\bar{r}}, \partial_{\bar{i}})$ and $(\delta^*_i, \partial_{\bar{i}}) = (\partial_i - N^{*r}_{\ i}\partial_{\bar{r}}, \partial_{\bar{i}})$ of $TM^\times$, and two coframes $(dx^i, \delta y^i) = (dx^i, dy^i + N^i_j dx^j)$ and $(dx^i, \delta^* y^i) = (dx^i, dy^i + N^{*i}_{\ j} dx^j)$, respectively.
We put

$$N^{*i}_{\ j} - N^i_j = B^i_j. \tag{3.2}$$

Then we have

$$\delta^*_i = \delta_i - B^r_i \partial_{\bar{r}}, \qquad \delta^* y^i = \delta y^i + B^i_r dx^r. \tag{3.3}$$

Further we have

$$\nabla_{\delta_j}\delta_i = F^r_{ij}\delta_r, \quad \nabla_{\delta_j}\partial_{\bar{i}} = F^r_{ij}\partial_{\bar{r}}, \quad \nabla_{\partial_{\bar{j}}}\delta_i = C^r_{ij}\delta_r, \quad \nabla_{\partial_{\bar{j}}}\partial_{\bar{i}} = C^r_{ij}\partial_{\bar{r}}, \tag{3.4}$$

$$\nabla_{\delta_j} dx^i = -F^i_{rj}dx^r, \ \nabla_{\delta_j}\delta y^i = -F^i_{rj}\delta y^r, \ \nabla_{\partial_{\bar{j}}} dx^i = -C^i_{rj}dx^r, \ \nabla_{\partial_{\bar{j}}}\delta y^i = -C^i_{rj}\delta y^r \tag{3.5}$$

and

$$\nabla^*_{\delta^*_j}\delta^*_i = F^{*r}_{\ ij}\delta^*_r, \quad \nabla^*_{\delta^*_j}\partial_{\bar{i}} = F^{*r}_{\ ij}\partial_{\bar{r}}, \quad \nabla^*_{\partial_{\bar{j}}}\delta^*_i = C^{*r}_{\ ij}\delta^*_r, \quad \nabla^*_{\partial_{\bar{j}}}\partial_{\bar{i}} = C^{*r}_{\ ij}\partial_{\bar{r}}, \tag{3.6}$$

(3.7)
$$\nabla^*_{\delta^*_j} dx^i = -F^{*i}_{\ rj}dx^r, \ \nabla^*_{\delta^*_j}\delta^* y^i = -F^{*i}_{\ rj}\delta^* y^r, \ \nabla^*_{\partial_{\bar{j}}} dx^i = -C^{*i}_{\ rj}dx^r, \ \nabla^*_{\partial_{\bar{j}}}\delta^* y^i = -C^{*i}_{\ rj}\delta^* y^r.$$

From (3.3) we also have

$$(3.8)\qquad \begin{aligned} &\nabla^*_{\delta_j}\delta_i = (F^{*p}_{ij} + C^{*p}_{ir}B^r_j)\delta_p + (B^p_{i|*j} + B^p_i|^*_r B^r_j)\partial_{\bar{p}},\\ &\nabla^*_{\delta_j}\partial_{\bar{i}} = (F^{*p}_{ij} + C^{*p}_{ir}B^r_j)\partial_{\bar{p}},\quad \nabla^*_{\partial_{\bar{j}}}\delta_i = C^{*p}_{ij}\delta_p + B^p_i|^*_j\partial_{\bar{p}},\quad \nabla^*_{\partial_{\bar{j}}}\partial_{\bar{i}} = C^{*p}_{ij}\partial_{\bar{p}}.\end{aligned}$$

In addition, from (3.8) we have for any vector fields $X = X^i\delta_i + X^{\bar{i}}\partial_{\bar{i}}$, $Y = Y^i\delta_i + Y^{\bar{i}}\partial_{\bar{i}}$ and $Z = Z^i\delta_i + Z^{\bar{i}}\partial_{\bar{i}}$ on $TM^\times$

$$(3.9)\qquad \begin{aligned} &\nabla_Z X = (Z^r X^i_{|r} + Z^{\bar{r}} X^i|_r)\delta_i + (Z^r X^{\bar{i}}_{|r} + Z^{\bar{r}} X^{\bar{i}}|_r)\partial_{\bar{i}},\\ &\nabla^*_Z Y = (Z^r(Y^i_{|*r} + B^l_r Y^i|^*_l) + Z^{\bar{r}} Y^i|^*_r)\delta_i\\ &+ (Z^r(Y^p(B^i_{p|*r} + B^l_r B^i_p|^*_l) + (Y^{\bar{i}}_{|*r} + B^l_r Y^{\bar{i}}|^*_l)) + Z^{\bar{r}}(Y^p B^i_p|^*_r + Y^{\bar{i}}|^*_r))\partial_{\bar{i}}.\end{aligned}$$

(I) The case of Vertical lift

Firstly, we consider the case of vertical lift of the Finsler metric $g_{ij}(x,y)$. We assume that

$$(3.10)\qquad Zg^v(X,Y) = g^v(\nabla_Z X, Y) + g^v(X, \nabla^*_Z Y).$$

From (2.1), the left hand side of (3.10) is

(3.11)

$$\begin{aligned} Zg^v(X,Y) &= Z^r\delta_r(g_{ij}X^iY^j) + Z^{\bar{r}}\partial_{\bar{r}}(g_{ij}X^iY^j)\\ &= Z^r(\delta_r g_{ij}X^iY^j + g_{ij}\delta_r X^iY^j + g_{ij}X^i\delta_r Y^j) + Z^{\bar{r}}(\partial_{\bar{r}} g_{ij}X^iY^j + g_{ij}\partial_{\bar{r}}X^iY^j + g_{ij}X^i\partial_{\bar{r}}Y^j).\end{aligned}$$

On the other hand, from the first equation of (3.9), the first term of the right hand side of (3.10) is

(3.12)

$$\begin{aligned} g^v(\nabla_Z X, Y) &= g_{ij}dx^i((Z^r X^p_{|r} + Z^{\bar{r}}X^p|_r)\delta_p + (Z^r X^{\bar{p}}_{|r} + Z^{\bar{r}}X^{\bar{p}}|_r)\partial_{\bar{p}})dx^j(Y^r\delta_r + Y^{\bar{r}}\partial_{\bar{r}})\\ &= g_{ij}(Z^r X^i_{|r} + Z^{\bar{r}}X^i|_r)Y^j\end{aligned}$$

and from the second equation of (3.9), the second term of the right hand side of (3.10) is

(3.13)

$$\begin{aligned} g^v(X, \nabla^*_Z Y) &= g_{ij}dx^i(X^r\delta_r + X^{\bar{r}}\partial_{\bar{r}})dx^j((Z^r(Y^p_{|*r} + B^l_r Y^p|^*_l) + Z^{\bar{r}}Y^p|^*_r)\delta_p\\ &\quad + (Z^r(Y^p(B^k_{p|*r} + B^l_r B^k_p|^*_l) + (Y^{\bar{k}}_{|*r} + B^l_r Y^{\bar{k}}|^*_l)) + Z^{\bar{r}}(Y^p B^k_p|^*_r + Y^{\bar{k}}|^*_r))\partial_{\bar{k}})\\ &= g_{ij}X^i(Z^r(Y^j_{|*r} + B^l_r Y^j|^*_l) + Z^{\bar{r}}Y^j|^*_r).\end{aligned}$$

Therefore, from (3.12) and (3.13), the right hand side of (3.10) satisfies as follows

(3.14)

$$\begin{aligned} &g^v(\nabla_Z X, Y) + g^v(X, \nabla^*_Z Y) = g_{ij}(Z^r X^i_{|r} + Z^{\bar{r}}X^i|_r)Y^j + g_{ij}X^i(Z^r(Y^j_{|*r} + B^l_r Y^j|^*_l) + Z^{\bar{r}}Y^j|^*_r)\\ &= Z^r(g_{ij}X^i_{|r}Y^j + g_{ij}X^iY^j_{|*r} + g_{ij}X^iB^l_rY^j|^*_l) + Z^{\bar{r}}(g_{ij}X^i|_rY^j + g_{ij}X^iY^j|^*_r).\end{aligned}$$

Finally, from (3.11), (3.14) and the arbitrariness of Z, we have

$$\delta_r g_{ij}X^iY^j + g_{ij}\delta_r X^iY^j + g_{ij}X^i\delta_r Y^j = g_{ij}X^i_{|r}Y^j + g_{ij}X^iY^j_{|^*r} + g_{ij}X^iB^l_rY^j|^*_l, \tag{3.15}$$

$$\partial_{\bar{r}} g_{ij}X^iY^j + g_{ij}\partial_{\bar{r}} X^iY^j + g_{ij}X^i\partial_{\bar{r}} Y^j = g_{ij}X^i|_rY^j + g_{ij}X^iY^j|^*_r. \tag{3.16}$$

Here we take the following relations

$$Y^j_{|^*r} = \delta^*_r Y^j + F^{*j}_{\ pr}Y^p = \delta_r Y^j - B^l_r\partial_{\bar{l}}Y^j + F^{*j}_{\ pr}Y^p \tag{3.17}$$

and notice $X^i_{|r} = \delta_r X^i + F^i_{pr}X^p$ and $Y^j|^*_l = \partial_{\bar{l}}Y^j + C^{*j}_{\ pl}Y^p$, then from (3.15) and (3.16) we have

$$\delta_r g_{ij}X^iY^j = (g_{kj}F^k_{ir} + g_{ik}F^{*k}_{\ jr} + g_{ik}C^{*k}_{\ jl}B^l_r)X^iY^j, \tag{3.18}$$

$$\partial_{\bar{r}} g_{ij}X^iY^j = (g_{kj}C^k_{ir} + g_{ik}C^{*k}_{\ jr})X^iY^j, \tag{3.19}$$

respectively. Since the arbitrariness of $X$and Y, we can obtain

$$\delta_r g_{ij} = g_{kj}F^k_{ir} + g_{ik}F^{*k}_{\ jr} + g_{ik}C^{*k}_{\ jl}B^l_r, \tag{3.20}$$

$$\partial_{\bar{r}} g_{ij} = g_{kj}C^k_{ir} + g_{ik}C^{*k}_{\ jr}. \tag{3.21}$$

Thus we have

Theorem 3.1 *Let $F\Gamma = (N, \nabla)$ be a Finsler connection and $F\Gamma^* = (N^*, \nabla^*)$ another one satisfying* (3.2) *on a Finsler space (M, F). If ∇, ∇^* and g^v satisfy the equation* (3.10), *then* (3.20) *and* (3.21) *are satisfied.*

(II) The case of Horizontal lift

Next we consider the case of horizontal lifts of the Finsler metric $g_{ij}(x, y)$. We assume that

$$Zg^h(X, Y) = g^h(\nabla_Z X, Y) + g^h(X, \nabla^*_Z Y). \tag{3.22}$$

From (2.2), the left hand side of (3.22) is

(3.23)

$$\begin{aligned} Zg^h(X,Y) &= Z^r\delta_r(g_{ij}X^{\bar{i}}Y^j + g_{ij}X^iY^{\bar{j}}) + Z^{\bar{r}}\partial_{\bar{r}}(g_{ij}X^{\bar{i}}Y^j + g_{ij}X^iY^{\bar{j}}) \\ &= Z^r(\delta_r g_{ij}X^{\bar{i}}Y^j + g_{ij}\delta_r X^{\bar{i}}Y^j + g_{ij}X^{\bar{i}}\delta_r Y^j + \delta_r g_{ij}X^iY^{\bar{j}} + g_{ij}\delta_r X^iY^{\bar{j}} + g_{ij}X^i\delta_r Y^{\bar{j}}) \\ &\quad + Z^{\bar{r}}(\partial_{\bar{r}} g_{ij}X^{\bar{i}}Y^j + g_{ij}\partial_{\bar{r}} X^{\bar{i}}Y^j + g_{ij}X^{\bar{i}}\partial_{\bar{r}} Y^j + \partial_{\bar{r}} g_{ij}X^iY^{\bar{j}} + g_{ij}\partial_{\bar{r}} X^iY^{\bar{j}} + g_{ij}X^i\partial_{\bar{r}} Y^{\bar{j}}). \end{aligned}$$

On the other hand, from the first equation of (3.9), the first term of the right hand side of (3.22) is

(3.24)

$$\begin{aligned} g^h(\nabla_Z X, Y) &= g_{ij}\delta y^i((Z^rX^p_{|r} + Z^{\bar{r}}X^p|_r)\delta_p + (Z^rX^{\bar{p}}_{|r} + Z^{\bar{r}}X^{\bar{p}}|_r)\partial_{\bar{p}})dx^j(Y^r\delta_r + Y^{\bar{r}}\partial_{\bar{r}}) \\ &\quad + g_{ij}dx^i((Z^rX^p_{|r} + Z^{\bar{r}}X^p|_r)\delta_p + (Z^rX^{\bar{p}}_{|r} + Z^{\bar{r}}X^{\bar{p}}|_r)\partial_{\bar{p}})\delta y^j(Y^r\delta_r + Y^{\bar{r}}\partial_{\bar{r}}) \\ &= g_{ij}(Z^rX^{\bar{i}}_{|r} + Z^{\bar{r}}X^{\bar{i}}|_r)Y^j + g_{ij}(Z^rX^i_{|r} + Z^{\bar{r}}X^i|_r)Y^{\bar{j}} \end{aligned}$$

and from the second equation of (3.9), the second term of the right hand side of (3.22) is

(3.25)

$$
\begin{aligned}
g^h(X, \nabla^*_Z Y) &= g_{ij}\delta y^i (X^r \delta_r + X^{\bar{r}}\partial_{\bar{r}}) dx^j ((Z^r (Y^p_{|*r} + B^l_r Y^p|^*_l) + Z^{\bar{r}} Y^p|^*_r)\delta_p \\
&+ (Z^r (Y^p (B^k_{p|*r} + B^l_r B^k_p|^*_l) + (Y^{\bar{k}}_{|*r} + B^l_r Y^{\bar{k}}|^*_l)) + Z^{\bar{r}} (Y^p B^k_p|^*_r + Y^{\bar{k}}|^*_r))\partial_{\bar{k}}) \\
&+ g_{ij} dx^i (X^r \delta_r + X^{\bar{r}}\partial_{\bar{r}}) \delta y^j ((Z^r (Y^p_{|*r} + B^l_r Y^p|^*_l) + Z^{\bar{r}} Y^p|^*_r)\delta_p \\
&+ (Z^r (Y^p (B^k_{p|*r} + B^l_r B^k_p|^*_l) + (Y^{\bar{k}}_{|*r} + B^l_r Y^{\bar{k}}|^*_l)) + Z^{\bar{r}} (Y^p B^k_p|^*_r + Y^{\bar{k}}|^*_r))\partial_{\bar{k}}) \\
&= g_{ij} X^{\bar{i}} (Z^r (Y^j_{|*r} + B^l_r Y^j|^*_l) + Z^{\bar{r}} Y^j|^*_r) \\
&+ g_{ij} X^i (Z^r (Y^p (B^j_{p|*r} + B^l_r B^j_p|^*_l) + (Y^{\bar{j}}_{|*r} + B^l_r Y^{\bar{j}}|^*_l)) + Z^{\bar{r}} (Y^p B^j_p|^*_r + Y^{\bar{j}}|^*_r)).
\end{aligned}
$$

Therefore, from (3.24) and (3.25), the right hand side of (3.22) satisfies as follows

(3.26)

$$
\begin{aligned}
&g^h(\nabla_Z X, Y) + g^h(X, \nabla^*_Z Y) = g_{ij}(Z^r X^{\bar{i}}_{|r} + Z^{\bar{r}} X^{\bar{i}}|_r) Y^j + g_{ij}(Z^r X^i_{|r} + Z^{\bar{r}} X^i|_r) Y^{\bar{j}} \\
&+ g_{ij} X^{\bar{i}} (Z^r (Y^j_{|*r} + B^l_r Y^j|^*_l) + Z^{\bar{r}} Y^j|^*_r) \\
&+ g_{ij} X^i (Z^r (Y^p (B^j_{p|*r} + B^l_r B^j_p|^*_l) + (Y^{\bar{j}}_{|*r} + B^l_r Y^{\bar{j}}|^*_l)) + Z^{\bar{r}} (Y^p B^j_p|^*_r + Y^{\bar{j}}|^*_r)) \\
&= Z^r (g_{ij} X^{\bar{i}}_{|r} Y^j + g_{ij} X^i_{|r} Y^{\bar{j}} + g_{ij} X^{\bar{i}} (Y^j_{|*r} + B^l_r Y^j|^*_l) + g_{ij} X^i (Y^p (B^j_{p|*r} + B^l_r B^j_p|^*_l) + (Y^{\bar{j}}_{|*r} + B^l_r Y^{\bar{j}}|^*_l))) \\
&+ Z^{\bar{r}} (g_{ij} X^{\bar{i}}|_r Y^j + g_{ij} X^i|_r Y^{\bar{j}} + g_{ij} X^{\bar{i}} Y^j|^*_r + g_{ij} X^i (Y^p B^j_p|^*_r + Y^{\bar{j}}|^*_r)).
\end{aligned}
$$

Finally, from (3.23), (3.26) and the arbitrariness of Z, we have

(3.27)

$$
\begin{aligned}
&\delta_r g_{ij} X^{\bar{i}} Y^j + g_{ij}\delta_r X^{\bar{i}} Y^j + g_{ij} X^{\bar{i}} \delta_r Y^j + \delta_r g_{ij} X^i Y^{\bar{j}} + g_{ij}\delta_r X^i Y^{\bar{j}} + g_{ij} X^i \delta_r Y^{\bar{j}} \\
&= g_{ij} X^{\bar{i}}_{|r} Y^j + g_{ij} X^i_{|r} Y^{\bar{j}} + g_{ij} X^{\bar{i}} (Y^j_{|*r} + B^l_r Y^j|^*_l) + g_{ij} X^i (Y^p (B^j_{p|*r} + B^l_r B^j_p|^*_l) + (Y^{\bar{j}}_{|*r} + B^l_r Y^{\bar{j}}|^*_l)),
\end{aligned}
$$

$$
(3.28) \quad
\begin{aligned}
&\partial_{\bar{r}} g_{ij} X^{\bar{i}} Y^j + g_{ij}\partial_{\bar{r}} X^{\bar{i}} Y^j + g_{ij} X^{\bar{i}} \partial_{\bar{r}} Y^j + \partial_{\bar{r}} g_{ij} X^i Y^{\bar{j}} + g_{ij}\partial_{\bar{r}} X^i Y^{\bar{j}} + g_{ij} X^i \partial_{\bar{r}} Y^{\bar{j}} \\
&= g_{ij} X^{\bar{i}}|_r Y^j + g_{ij} X^i|_r Y^{\bar{j}} + g_{ij} X^{\bar{i}} Y^j|^*_r + g_{ij} X^i (Y^p B^j_p|^*_r + Y^{\bar{j}}|^*_r).
\end{aligned}
$$

Here we take the following relations

$$
(3.29) \qquad Y^{\bar{j}}_{|*r} = \delta^*_r Y^{\bar{j}} + F^{*j}_{\ \ pr} Y^{\bar{p}} = \delta_r Y^{\bar{j}} - B^l_r \partial_{\bar{l}} Y^{\bar{j}} + F^{*j}_{\ \ pr} Y^{\bar{p}}
$$

and notice (3.17) and $Y^{\bar{j}}|^*_l = \partial_{\bar{l}} Y^{\bar{j}} + C^{*j}_{\ \ pl} Y^{\bar{p}}$, then from (3.27) we have

(3.30)

$$
\delta_r g_{ij} (X^{\bar{i}} Y^j + X^i Y^{\bar{j}}) = (g_{kj} F^k_{ir} + g_{ik} F^{*k}_{\ \ jr} + g_{ik} C^{*k}_{\ \ jl} B^l_r)(X^{\bar{i}} Y^j + X^i Y^{\bar{j}}) + g_{ij} X^i Y^k (B^j_{k|*r} + B^l_r B^j_k|^*_l).
$$

Since the arbitrariness of $X$and Y, we can obtain

$$
(3.31) \qquad \delta_r g_{ij} = g_{kj} F^k_{ir} + g_{ik} F^{*k}_{\ \ jr} + g_{ik} C^{*k}_{\ \ jl} B^l_r,
$$

$$
(3.32) \qquad B^j_{k|*r} + B^l_r B^j_k|^*_l = 0.
$$

By the same way, from (3.28) we have

$$\partial_{\bar{r}} g_{ij}(X^{\bar{i}}Y^j + X^iY^{\bar{j}}) = (g_{kj}C^k_{ir} + g_{ik}C^{*k}_{jr})(X^{\bar{i}}Y^j + X^iY^{\bar{j}}) + g_{ij}X^iY^kB^j_k|^*_r. \tag{3.33}$$

Since the arbitrariness of $X$and Y, we can obtain

$$\partial_{\bar{r}} g_{ij} = g_{kj}C^k_{ir} + g_{ik}C^{*k}_{jr}, \tag{3.34}$$

$$B^j_k|^*_r = 0. \tag{3.35}$$

From (3.32) and (3.35),

$$B^j_{p|^*r} = 0, \ B^j_p|^*_r = 0 \tag{3.36}$$

are satisfied. Thus we have

Theorem 3.2 *Let* $F\Gamma = (N, \nabla)$ *be a Finsler connection and* $F\Gamma^* = (N^*, \nabla^*)$ *another one satisfying* (3.2) *on a Finsler space* (M, F). *If* ∇, ∇^* *and* g^h *satisfy the equation* (3.22), *then* (3.31), (3.34) *and* (3.36) *are satisfied.*

(III) The case of Complete lift

Next we consider the case of Complete lifts of the Finsler metric $g_{ij}(x, y)$. We assume that

$$Zg^c(X, Y) = g^c(\nabla_Z X, Y) + g^c(X, \nabla^*_Z Y). \tag{3.37}$$

From (2.3), the left hand side of (3.37) is

(3.38)

$$\begin{aligned}
Zg^c(X,Y) &= Z^r\delta_r(g_{ij|0}X^iY^j + g_{ij}X^{\bar{i}}Y^j + g_{ij}X^iY^{\bar{j}}) + Z^{\bar{r}}\partial_{\bar{r}}(g_{ij|0}X^iY^j + g_{ij}X^{\bar{i}}Y^j + g_{ij}X^iY^{\bar{j}}) \\
&= Z^r(\delta_r g_{ij|0}X^iY^j + g_{ij|0}\delta_r X^iY^j + g_{ij|0}X^i\delta_r Y^j + \delta_r g_{ij}X^{\bar{i}}Y^j + g_{ij}\delta_r X^{\bar{i}}Y^j + g_{ij}X^{\bar{i}}\delta_r Y^j \\
&+ \delta_r g_{ij}X^iY^{\bar{j}} + g_{ij}\delta_r X^iY^{\bar{j}} + g_{ij}X^i\delta_r Y^{\bar{j}}) \\
&+ Z^{\bar{r}}(\partial_{\bar{r}} g_{ij|0}X^iY^j + g_{ij|0}\partial_{\bar{r}} X^iY^j + g_{ij|0}X^i\partial_{\bar{r}} Y^j + \partial_{\bar{r}} g_{ij}X^{\bar{i}}Y^j + g_{ij}\partial_{\bar{r}} X^{\bar{i}}Y^j + g_{ij}X^{\bar{i}}\partial_{\bar{r}} Y^j \\
&+ \partial_{\bar{r}} g_{ij}X^iY^{\bar{j}} + g_{ij}\partial_{\bar{r}} X^iY^{\bar{j}} + g_{ij}X^i\partial_{\bar{r}} Y^{\bar{j}}).
\end{aligned}$$

On the other hand, from the first equation of (3.9), the first term of the right hand side of (3.37) is

(3.39)

$$\begin{aligned}
g^c(\nabla_Z X, Y) &= g_{ij|0}dx^i((Z^rX^p_{|r} + Z^{\bar{r}}X^p|_r)\delta_p + (Z^rX^{\bar{p}}_{|r} + Z^{\bar{r}}X^{\bar{p}}|_r)\partial_{\bar{p}})dx^j(Y^r\delta_r + Y^{\bar{r}}\partial_{\bar{r}}) \\
&+ g_{ij}\delta y^i((Z^rX^p_{|r} + Z^{\bar{r}}X^p|_r)\delta_p + (Z^rX^{\bar{p}}_{|r} + Z^{\bar{r}}X^{\bar{p}}|_r)\partial_{\bar{p}})dx^j(Y^r\delta_r + Y^{\bar{r}}\partial_{\bar{r}}) \\
&+ g_{ij}dx^i((Z^rX^p_{|r} + Z^{\bar{r}}X^p|_r)\delta_p + (Z^rX^{\bar{p}}_{|r} + Z^{\bar{r}}X^{\bar{p}}|_r)\partial_{\bar{p}})\delta y^j(Y^r\delta_r + Y^{\bar{r}}\partial_{\bar{r}}) \\
&= g_{ij|0}(Z^rX^i_{|r} + Z^{\bar{r}}X^i|_r)Y^j + g_{ij}(Z^rX^{\bar{i}}_{|r} + Z^{\bar{r}}X^{\bar{i}}|_r)Y^j + g_{ij}(Z^rX^i_{|r} + Z^{\bar{r}}X^i|_r)Y^{\bar{j}}
\end{aligned}$$

and from the second equation of (3.9), the second term of the right hand side of (3.37) is

$$
\begin{aligned}
g^c(X, \nabla^*_Z Y) &= g_{ij|0} dx^i (X^r \delta_r + X^{\bar{r}} \partial_{\bar{r}}) dx^j ((Z^r (Y^p_{|*r} + B^l_r Y^p|^*_l) + Z^{\bar{r}} Y^p|^*_r)\delta_p \\
&+ (Z^r (Y^p (B^k_{p|*r} + B^l_r B^k_p|^*_l) + (Y^{\bar{k}}_{|*r} + B^l_r Y^{\bar{k}}|^*_l)) + Z^{\bar{r}} (Y^p B^k_p|^*_r + Y^{\bar{k}}|^*_r))\partial_{\bar{k}}) \\
&+ g_{ij} \delta y^i (X^r \delta_r + X^{\bar{r}} \partial_{\bar{r}}) dx^j ((Z^r (Y^p_{|*r} + B^l_r Y^p|^*_l) + Z^{\bar{r}} Y^p|^*_r)\delta_p \\
&+ (Z^r (Y^p (B^k_{p|*r} + B^l_r B^k_p|^*_l) + (Y^{\bar{k}}_{|*r} + B^l_r Y^{\bar{k}}|^*_l)) + Z^{\bar{r}} (Y^p B^k_p|^*_r + Y^{\bar{k}}|^*_r))\partial_{\bar{k}}) \\
&+ g_{ij} dx^i (X^r \delta_r + X^{\bar{r}} \partial_{\bar{r}}) \delta y^j ((Z^r (Y^p_{|*r} + B^l_r Y^p|^*_l) + Z^{\bar{r}} Y^p|^*_r)\delta_p \\
&+ (Z^r (Y^p (B^k_{p|*r} + B^l_r B^k_p|^*_l) + (Y^{\bar{k}}_{|*r} + B^l_r Y^{\bar{k}}|^*_l)) + Z^{\bar{r}} (Y^p B^k_p|^*_r + Y^{\bar{k}}|^*_r))\partial_{\bar{k}}) \\
&= g_{ij|0} X^i (Z^r (Y^j_{|*r} + B^l_r Y^j|^*_l) + Z^{\bar{r}} Y^j|^*_r) + g_{ij} X^{\bar{i}} (Z^r (Y^j_{|*r} + B^l_r Y^j|^*_l) + Z^{\bar{r}} Y^j|^*_r) \\
&+ g_{ij} X^i (Z^r (Y^p (B^j_{p|*r} + B^l_r B^j_p|^*_l) + (Y^{\bar{j}}_{|*r} + B^l_r Y^{\bar{j}}|^*_l)) + Z^{\bar{r}} (Y^p B^j_p|^*_r + Y^{\bar{j}}|^*_r)).
\end{aligned} \tag{3.40}
$$

Therefore, from (3.39) and (3.40), the right hand side of (3.37) satisfies as follows

(3.41)

$$
\begin{aligned}
& g^c(\nabla_Z X, Y) + g^c(X, \nabla^*_Z Y) \\
&= g_{ij|0} (Z^r X^i_{|r} + Z^{\bar{r}} X^i|_r) Y^j + g_{ij} (Z^r X^{\bar{i}}_{|r} + Z^{\bar{r}} X^{\bar{i}}|_r) Y^j + g_{ij} (Z^r X^i_{|r} + Z^{\bar{r}} X^i|_r) Y^{\bar{j}} \\
&+ g_{ij|0} X^i (Z^r (Y^j_{|*r} + B^l_r Y^j|^*_l) + Z^{\bar{r}} Y^j|^*_r) + g_{ij} X^{\bar{i}} (Z^r (Y^j_{|*r} + B^l_r Y^j|^*_l) + Z^{\bar{r}} Y^j|^*_r) \\
&+ g_{ij} X^i (Z^r (Y^p (B^j_{p|*r} + B^l_r B^j_p|^*_l) + (Y^{\bar{j}}_{|*r} + B^l_r Y^{\bar{j}}|^*_l)) + Z^{\bar{r}} (Y^p B^j_p|^*_r + Y^{\bar{j}}|^*_r)) \\
&= Z^r (g_{ij|0} X^i_{|r} Y^j + g_{ij|0} X^i Y^j_{|*r} + g_{ij|0} X^i B^l_r Y^j|^*_l + g_{ij} X^{\bar{i}}_{|r} Y^j + g_{ij} X^i_{|r} Y^{\bar{j}} + g_{ij} X^{\bar{i}} (Y^j_{|*r} + B^l_r Y^j|^*_l) \\
&+ g_{ij} X^i (Y^p (B^j_{p|*r} + B^l_r B^j_p|^*_l) + (Y^{\bar{j}}_{|*r} + B^l_r Y^{\bar{j}}|^*_l))) \\
&+ Z^{\bar{r}} (g_{ij|0} X^i|_r Y^j + g_{ij|0} X^i Y^j|^*_r + g_{ij} X^{\bar{i}}|_r Y^j + g_{ij} X^i|_r Y^{\bar{j}} + g_{ij} X^{\bar{i}} Y^j|^*_r + g_{ij} X^i (Y^p B^j_p|^*_r + Y^{\bar{j}}|^*_r)).
\end{aligned}
$$

Finally, from (3.38), (3.41) and the arbitrariness of Z, we have

(3.42)

$$
\begin{aligned}
& \delta_r g_{ij|0} X^i Y^j + g_{ij|0} \delta_r X^i Y^j + g_{ij|0} X^i \delta_r Y^j + \delta_r g_{ij} X^{\bar{i}} Y^j + g_{ij} \delta_r X^{\bar{i}} Y^j + g_{ij} X^{\bar{i}} \delta_r Y^j \\
&+ \delta_r g_{ij} X^i Y^{\bar{j}} + g_{ij} \delta_r X^i Y^{\bar{j}} + g_{ij} X^i \delta_r Y^{\bar{j}} \\
&= g_{ij|0} X^i_{|r} Y^j + g_{ij|0} X^i Y^j_{|*r} + g_{ij|0} X^i B^l_r Y^j|^*_l + g_{ij} X^{\bar{i}}_{|r} Y^j + g_{ij} X^i_{|r} Y^{\bar{j}} + g_{ij} X^{\bar{i}} (Y^j_{|*r} + B^l_r Y^j|^*_l) \\
&+ g_{ij} X^i (Y^p (B^j_{p|*r} + B^l_r B^j_p|^*_l) + (Y^{\bar{j}}_{|*r} + B^l_r Y^{\bar{j}}|^*_l),
\end{aligned}
$$

(3.43)

$$
\begin{aligned}
& \partial_{\bar{r}} g_{ij|0} X^i Y^j + g_{ij|0} \partial_{\bar{r}} X^i Y^j + g_{ij|0} X^i \partial_{\bar{r}} Y^j + \partial_{\bar{r}} g_{ij} X^{\bar{i}} Y^j + g_{ij} \partial_{\bar{r}} X^{\bar{i}} Y^j + g_{ij} X^{\bar{i}} \partial_{\bar{r}} Y^j \\
&+ \partial_{\bar{r}} g_{ij} X^i Y^{\bar{j}} + g_{ij} \partial_{\bar{r}} X^i Y^{\bar{j}} + g_{ij} X^i \partial_{\bar{r}} Y^{\bar{j}} \\
&= g_{ij|0} X^i|_r Y^j + g_{ij|0} X^i Y^j|^*_r + g_{ij} X^{\bar{i}}|_r Y^j + g_{ij} X^i|_r Y^{\bar{j}} + g_{ij} X^{\bar{i}} Y^j|^*_r + g_{ij} X^i (Y^p B^j_p|^*_r + Y^{\bar{j}}|^*_r).
\end{aligned}
$$

From (3.17), (3.29) and (3.42) we have

$$
\begin{aligned}
& \delta_r g_{ij|0} X^i Y^j + \delta_r g_{ij} (X^{\bar{i}} Y^j + X^i Y^{\bar{j}}) \\
&= (g_{kj|0} F^k_{ir} + g_{ik|0} F^{*k}_{jr} + g_{ik|0} B^l_r C^{*k}_{jl} + g_{ik} (B^k_{j|*r} + B^l_r B^k_j|^*_l)) X^i Y^j \\
&+ (g_{kj} F^k_{ir} + g_{ik} F^{*k}_{jr} + g_{ik} C^{*k}_{jl} B^l_r)(X^{\bar{i}} Y^j + X^i Y^{\bar{j}}).
\end{aligned} \tag{3.44}
$$

Since the arbitrariness of $X$and Y, we can obtain

$$\delta_r g_{ij|0} = g_{kj|0}F^k_{ir} + g_{ik|0}F^{*k}_{jr} + g_{ik|0}B^l_r C^{*k}_{jl} + g_{ik}(B^k_{j|*r} + B^l_r B^k_j|^*_l), \tag{3.45}$$
$$\delta_r g_{ij} = g_{kj}F^k_{ir} + g_{ik}F^{*k}_{jr} + g_{ik}C^{*k}_{jl}B^l_r. \tag{3.46}$$

Further from (3.17), (3.29) and (3.43) we have

$$\begin{aligned}&\partial_{\bar r} g_{ij|0}X^iY^j + \partial_{\bar r} g_{ij}(X^{\bar i}Y^j + X^iY^{\bar j})\\ &= (g_{kj|0}C^k_{ir} + g_{ik|0}C^{*k}_{jr} + g_{ik}B^k_j|^*_l)X^iY^j + (g_{kj}C^k_{ir} + g_{ik}C^{*k}_{jr})(X^{\bar i}Y^j + X^iY^{\bar j}).\end{aligned} \tag{3.47}$$

Since the arbitrariness of $X$and Y, we can obtain

$$\partial_{\bar r} g_{ij|0} = g_{kj|0}C^k_{ir} + g_{ik|0}C^{*k}_{jr} + g_{ik}B^k_j|^*_l, \tag{3.48}$$
$$\partial_{\bar r} g_{ij} = g_{kj}C^k_{ir} + g_{ik}C^{*k}_{jr}. \tag{3.49}$$

Thus we have

Theorem 3.3 *Let $F\Gamma = (N, \nabla)$ be a Finsler connection and $F\Gamma^* = (N^*, \nabla^*)$ another one satisfying* (3.2) *on a Finsler space (M, F). If ∇, ∇^* and g^c satisfy the equation* (3.37), *then* (3.45), (3.46) *and* (3.48), (3.49) *are satisfied.*

(IV) The case of Sasaki lift

Lastly, we consider the case of Sasaki lifts of the Finsler metric $g_{ij}(x, y)$. We assume that

$$Zg^s(X, Y) = g^s(\nabla_Z X, Y) + g^s(X, \nabla^*_Z Y). \tag{3.50}$$

From (2.4), the left hand side of (3.50) is

(3.51)

$$\begin{aligned}Zg^s(X, Y) &= Z^r\delta_r(g_{ij}X^iY^j + g_{ij}X^{\bar i}Y^{\bar j}) + Z^{\bar r}\partial_{\bar r}(g_{ij}X^iY^j + g_{ij}X^{\bar i}Y^{\bar i})\\ &= Z^r(\delta_r g_{ij}X^iY^j + g_{ij}\delta_r X^iY^j + g_{ij}X^i\delta_r Y^j + \delta_r g_{ij}X^{\bar i}Y^{\bar j} + g_{ij}\delta_r X^{\bar i}Y^{\bar j} + g_{ij}X^{\bar i}\delta_r Y^{\bar j})\\ &+ Z^{\bar r}(\partial_{\bar r} g_{ij}X^iY^j + g_{ij}\partial_{\bar r} X^iY^j + g_{ij}X^i\partial_{\bar r} Y^j + \partial_{\bar r} g_{ij}X^{\bar i}Y^{\bar i} + g_{ij}\partial_{\bar r} X^{\bar i}Y^{\bar i} + g_{ij}X^{\bar i}\partial_{\bar r} Y^{\bar i}).\end{aligned}$$

On the other hand, from the first equation of (3.9), the first term of the right hand side of (3.50) is

(3.52)

$$\begin{aligned}&g^s(\nabla_Z X, Y) = g_{ij}dx^i((Z^rX^p_{|r} + Z^{\bar r}X^p|_r)\delta_p + (Z^rX^{\bar p}_{|r} + Z^{\bar r}X^{\bar p}|_r)\partial_{\bar p})dx^j(Y^r\delta_r + Y^{\bar r}\partial_{\bar r})\\ &+ g_{ij}\delta y^i((Z^rX^p_{|r} + Z^{\bar r}X^p|_r)\delta_p + (Z^rX^{\bar p}_{|r} + Z^{\bar r}X^{\bar p}|_r)\partial_{\bar p})\delta y^j(Y^r\delta_r + Y^{\bar r}\partial_{\bar r})\\ &= g_{ij}(Z^rX^i_{|r} + Z^{\bar r}X^i|_r)Y^j + g_{ij}(Z^rX^{\bar i}_{|r} + Z^{\bar r}X^{\bar i}|_r)Y^{\bar j}\end{aligned}$$

and from the second equation of (3.9), the second term of the right hand side of (3.50) is

$$(3.53)\quad \begin{aligned} g^s(X,\nabla^*_Z Y) &= g_{ij}dx^i(X^r\delta_r + X^{\bar r}\partial_{\bar r})dx^j((Z^r(Y^p_{|*r} + B^l_r Y^p|^*_l) + Z^{\bar r}Y^p|^*_r)\delta_p \\ &+ (Z^r(Y^p(B^{\bar k}_{p|*r} + B^l_r B^{\bar k}_p|^*_l) + (Y^{\bar k}_{|*r} + B^l_r Y^{\bar k}|^*_l)) + Z^{\bar r}(Y^p B^{\bar k}_p|^*_r + Y^{\bar k}|^*_r))\partial_{\bar k}) \\ &+ g_{ij}\delta y^i(X^r\delta_r + X^{\bar r}\partial_{\bar r})\delta y^j((Z^r(Y^p_{|*r} + B^l_r Y^p|^*_l) + Z^{\bar r}Y^p|^*_r)\delta_p \\ &+ (Z^r(Y^p(B^{\bar k}_{p|*r} + B^l_r B^{\bar k}_p|^*_l) + (Y^{\bar k}_{|*r} + B^l_r Y^{\bar k}|^*_l)) + Z^{\bar r}(Y^p B^{\bar k}_p|^*_r + Y^{\bar k}|^*_r))\partial_{\bar k}) \\ &= g_{ij}X^i(Z^r(Y^j_{|*r} + B^l_r Y^j|^*_l) + Z^{\bar r}Y^j|^*_r) \\ &+ g_{ij}X^{\bar i}(Z^r(Y^p(B^{\bar j}_{p|*r} + B^l_r B^{\bar j}_p|^*_l) + (Y^{\bar j}_{|*r} + B^l_r Y^{\bar j}|^*_l)) + Z^{\bar r}(Y^p B^{\bar j}_p|^*_r + Y^{\bar j}|^*_r)). \end{aligned}$$

Therefore, from (3.52) and (3.53), the right hand side of (3.50) satisfies as follows

$$(3.54)\quad \begin{aligned} & g^s(\nabla_Z X, Y) + g^s(X,\nabla^*_Z Y) \\ &= g_{ij}(Z^r X^i_{|r} + Z^{\bar r}X^i|_r)Y^j + g_{ij}(Z^r X^{\bar i}_{|r} + Z^{\bar r}X^{\bar i}|_r)Y^{\bar j} \\ &+ g_{ij}X^i(Z^r(Y^j_{|*r} + B^l_r Y^j|^*_l) + Z^{\bar r}Y^j|^*_r) \\ &+ g_{ij}X^{\bar i}(Z^r(Y^p(B^{\bar j}_{p|*r} + B^l_r B^{\bar j}_p|^*_l) + (Y^{\bar j}_{|*r} + B^l_r Y^{\bar j}|^*_l)) + Z^{\bar r}(Y^p B^{\bar j}_p|^*_r + Y^{\bar j}|^*_r)) \\ &= Z^r(g_{ij}X^i_{|r}Y^j + g_{ij}X^{\bar i}_{|r}Y^{\bar j} + g_{ij}X^i Y^j_{|*r} + g_{ij}X^i B^l_r Y^j|^*_l \\ &+ g_{ij}X^{\bar i}(Y^p(B^{\bar j}_{p|*r} + B^l_r B^{\bar j}_p|^*_l) + (Y^{\bar j}_{|*r} + B^l_r Y^{\bar j}|^*_l))) \\ &+ Z^{\bar r}(g_{ij}X^i|_r Y^j + g_{ij}X^{\bar i}|_r Y^{\bar j} + g_{ij}X^i Y^j|^*_l + g_{ij}X^{\bar i}(Y^p B^{\bar j}_p|^*_r + Y^{\bar j}|^*_r)). \end{aligned}$$

Finally, from (3.51), (3.54) and the arbitrariness of Z, we have

$$(3.55)\quad \begin{aligned} & \delta_r g_{ij}X^i Y^j + g_{ij}\delta_r X^i Y^j + g_{ij}X^i\delta_r Y^j + \delta_r g_{ij}X^{\bar i}Y^{\bar j} + g_{ij}\delta_r X^{\bar i}Y^{\bar j} + g_{ij}X^{\bar i}\delta_r Y^{\bar j} \\ &= g_{ij}X^i_{|r}Y^j + g_{ij}X^{\bar i}_{|r}Y^{\bar j} + g_{ij}X^i Y^j_{|*r} + g_{ij}X^i B^l_r Y^j|^*_l \\ &+ g_{ij}X^{\bar i}(Y^p(B^{\bar j}_{p|*r} + B^l_r B^{\bar j}_p|^*_l) + (Y^{\bar j}_{|*r} + B^l_r Y^{\bar j}|^*_l)), \end{aligned}$$

$$(3.56)\quad \begin{aligned} & \partial_{\bar r} g_{ij}X^i Y^j + g_{ij}\partial_{\bar r}X^i Y^j + g_{ij}X^i\partial_{\bar r}Y^j + \partial_{\bar r}g_{ij}X^{\bar i}Y^{\bar i} + g_{ij}\partial_{\bar r}X^{\bar i}Y^{\bar i} + g_{ij}X^{\bar i}\partial_{\bar r}Y^{\bar i} \\ &= g_{ij}X^i|_r Y^j + g_{ij}X^{\bar i}|_r Y^{\bar j} + g_{ij}X^i Y^j|^*_l + g_{ij}X^{\bar i}(Y^p B^{\bar j}_p|^*_r + Y^{\bar j}|^*_r). \end{aligned}$$

From (3.17), (3.29) and (3.55) we have

$$(3.57)\quad \begin{aligned} & \delta_r g_{ij}(X^i Y^j + X^{\bar i}Y^{\bar j}) \\ &= (g_{kj}F^k_{ir} + g_{ik}F^{*k}_{\ jr} + g_{ik}B^l_r C^{*k}_{\ jl})(X^i Y^j + X^{\bar i}Y^{\bar j}) + g_{ij}(B^{\bar j}_{p|*r} + B^l_r B^{\bar j}_p|^*_l)X^{\bar i}Y^p. \end{aligned}$$

Since the arbitrariness of $X$and Y, we can obtain

$$(3.58)\quad \delta_r g_{ij} = g_{kj}F^k_{ir} + g_{ik}F^{*k}_{\ jr} + g_{ik}B^l_r C^{*k}_{\ jl},$$

$$(3.59)\quad B^{\bar j}_{p|*r} + B^l_r B^{\bar j}_p|^*_l = 0.$$

Further from (3.17), (3.29) and (3.56) we have

$$(3.60)\quad \begin{aligned} & \partial_{\bar r} g_{ij}(X^i Y^j + X^{\bar i}Y^{\bar j}) \\ &= (g_{kj}C^k_{ir} + g_{ik}C^{*k}_{\ jr})(X^i Y^j + X^{\bar i}Y^{\bar j}) + g_{ij}X^{\bar i}Y^p B^{\bar j}_p|^*_r. \end{aligned}$$

Since the arbitrariness of $X$and Y, we can obtain

$$\partial_{\bar{r}} g_{ij} = g_{kj} C^k_{ir} + g_{ik} C^{*k}_{jr}, \tag{3.61}$$

$$B^j_p|^*_r = 0. \tag{3.62}$$

From (3.59) and (3.62),

$$B^j_{p|^*r} = 0, \ B^j_p|^*_r = 0 \tag{3.63}$$

are satisfied. Thus we have

Theorem 3.4 *Let $F\Gamma = (N, \nabla)$ be a Finsler connection and $F\Gamma^* = (N^*, \nabla^*)$ another one satisfying* (3.2) *on a Finsler space (M, F). If ∇, ∇^* and g^s satisfy the equation* (3.50), *then* (3.58), (3.61) *and* (3.63) *are satisfied.*

By the way, we can consider the case of $N = N^*$. Then we have $B^i_j = 0$. Therefore we also have the following corollaries.

Corollary 3.1 *Let $F\Gamma = (N, \nabla) = (N^i_j, F^i_{jk}, C^i_{jk})$ be a Finsler connection and $F\Gamma^* = (N, \nabla^*) = (N^i_j, F^{*i}_{jk}, C^{*i}_{jk})$ another one on a Finsler space (M, F). If ∇, ∇^* and g^v satisfy the equation* (3.10), *then $\delta_r g_{ij} = g_{kj} F^k_{ir} + g_{ik} F^{*k}_{jr}$ and $\partial_{\bar{r}} g_{ij} = g_{kj} C^k_{ir} + g_{ik} C^{*k}_{jr}$ are satisfied.*

Corollary 3.2 *Let $F\Gamma = (N, \nabla) = (N^i_j, F^i_{jk}, C^i_{jk})$ be a Finsler connection and $F\Gamma^* = (N, \nabla^*) = (N^i_j, F^{*i}_{jk}, C^{*i}_{jk})$ another one on a Finsler space (M, F). If ∇, ∇^* and g^h satisfy the equation* (3.22), *then $\delta_r g_{ij} = g_{kj} F^k_{ir} + g_{ik} F^{*k}_{jr}$ and $\partial_{\bar{r}} g_{ij} = g_{kj} C^k_{ir} + g_{ik} C^{*k}_{jr}$ are satisfied.*

Corollary 3.3 *Let $F\Gamma = (N, \nabla) = (N^i_j, F^i_{jk}, C^i_{jk})$ be a Finsler connection and $F\Gamma^* = (N, \nabla^*) = (N^i_j, F^{*i}_{jk}, C^{*i}_{jk})$ another one on a Finsler space (M, F). If ∇, ∇^* and g^c satisfy the equation* (3.37), *then $\delta_r g_{ij} = g_{kj} F^k_{ir} + g_{ik} F^{*k}_{jr}$ and $\partial_{\bar{r}} g_{ij} = g_{kj} C^k_{ir} + g_{ik} C^{*k}_{jr}$ are satisfied.*

Corollary 3.4 *Let $F\Gamma = (N, \nabla) = (N^i_j, F^i_{jk}, C^i_{jk})$ be a Finsler connection and $F\Gamma^* = (N, \nabla^*) = (N^i_j, F^{*i}_{jk}, C^{*i}_{jk})$ another one on a Finsler space (M, F). If ∇, ∇^* and g^s satisfy the equation* (3.50), *then $\delta_r g_{ij} = g_{kj} F^k_{ir} + g_{ik} F^{*k}_{jr}$ and $\partial_{\bar{r}} g_{ij} = g_{kj} C^k_{ir} + g_{ik} C^{*k}_{jr}$ are satisfied.*

Remark 3.1 *In the previous paper* [N05], *we see that the symmetric property of the linear connection ∇ consisted of the Finsler connection is the obstacle for Finsler spaces. The necessary of the symmetric property is in the Levi-Civita process to obtain coefficients of the connection. The key equation in its process is $\delta_r g_{ij} = g_{kj} F^k_{ir} + g_{ik} F^{*k}_{jr}$ or $\partial_{\bar{r}} g_{ij} = g_{kj} C^k_{ir} + g_{ik} C^{*k}_{jr}$. Therefore we can see the necessary conditions for the Finsler connection $F\Gamma = (N^i_j, F^i_{jk}, C^i_{jk})$ that $F^i_{jk} = F^i_{kj}$, namely, $T^i_{jk} = 0$ and $C^i_{jk} = C^i_{kj}$, namely, $S^i_{jk} = 0$.*

References

[A-K05] T. Aikou and L. Kozma : *Global Aspects of Finsler Geometry*, preprint, 2005.

[A-N00] S. Amari and H. Nagaoka : *Methods of Information Geometry*, Amer. Math. Soc., Oxford Univ. Press, 2000.

[M-I03] H. Matsuzoe and J. Inoguchi : *Statistical structures on tangent bundles*, Applied Sciences **5-1** (2003), pp.55-75.

[M86] M. Matsumoto : *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha, Press, Otsu, Japan, 1986.

[N05] T. Nagano : *On Statistical Structures and the lifts of Finsler Metrics*, Journal of the Faculty of Global Communication Siebold University of Nagasaki **6**(2005), pp191-197.

[Y-I73] K. Yano and S. Ishihara : *Tangent and Cotangent Bundles*, Marcel Dekker, 1973.