On the Geometry of Singular Finsler Spaces (Survey)

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Abstract

Thesis has:
Preface, Introduction, six Chapters and References.
The title of chapters
1. Preliminaries: Finsler, Lagrange and generalized Lagrange spaces.
2. The notion of Singular Finsler Spaces.
3. Singular Randers Spaces.
4. Variational Problem in the Singular Lagrange Spaces.
5. On the Connections of singular Finsler spaces.
6. Generalized Singular Finsler Spaces.

In Preface I described the history of the subject and the geometricians who worked in this field and the main problems which must be solved. Therefore the abstract is as follows, which numbers in [ ] and ( ) are refered to the P.h.D Thesis:

Preface

The notion of singular Finsler space was not defined till now. This is very clear in the case of singular Riemannian spaces. In this respect some remarkable papers were published by: Gr.Moisil, V.Oproiu etc.[58,72]. Other aspects of the partial degenerate Finsler spaces were studied in the paper [Atanasiu[2]].

In this Ph.D.Thesis we define the concept of singular Finsler space, as a natural extension of singular Riemannian space. We study the variational problem of the (nonregular) Lagrangian defined as the square of the fundamental function F of the space SF^n and the law of conservation of the energy of space SF^n. The theory is applied in the case of geodesics of mentioned space. We prove the existence of the spaces SF^n and give some examples like singular Randers spaces. The generalized singular Finsler spaces are introduced and studied, too.

The Lagrange spaces were introduced and studied by J. Kern[37] and R.Miron[47,48,49] in order to geometrize a fundamental concept in Analytical Mechanics. A Lagrange space $L^n = (M, L(x,y))$ is defined as a pair which consists of a real, smooth $n-$dimensional manifold M and a regular Lagrangian $L : TM \longrightarrow R$. It comes out that a Finsler space is a Lagrange space, but not conversely since the Lagrangian L may be not homogeneous with the respect to the variables (y^i), $i = 1, 2, \cdots, n$.

The fact that the Finsler spaces are particular Lagrange spaces suggested the developing of the geometry of the Lagrange spaces by extending the methods which have been used in the study of the geometry of Finsler spaces. In this way one can study sufficiently general regular Lagrangians which appear in mechanics, electrodynamics, optimal control etc.

The geometry of Lagrange spaces gives a model for both the gravitational and the electromagnetic field in a very natural blending of the geometrical structure of the space with the characteristic properties of these physical fields. This is possible due to of the utilization of some specific Lagrangians together with some fundamental concepts from the geometry of the total space TM of the tangent bundle (TM, π, M) as is for instance, the Liouville vector field, tangent structure etc.

As is expected, the variational problem formulated for the action integral of the regular Lagrangian $L(x,y)$ of a space L^n leads to the Euler-Lagrange equations which are very useful in the geometry of L^n. First, these are used in introducing a canonical connection and then a canonical metrical $d-$connection.

These two connections are basic in the geometry of L^n. Let us notice the considered notions capture both the symplectic structure induced by L^n on the manifold TM and the metrical structure on TM. These give together an almost Kählerian space K^{2n} determined on TM. It is the geometrical model of the Lagrange space L^n. It gives a geometrical legitimacy to the whole study of the Lagrange space L^n. We remark that the variational problem is applied also to the singular (non-regular) Lagrangians. This idea is very important in my study on the singular Finsler spaces.

The Finsler spaces with the fundamental function $F(x,y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i$, $(x,y) \in \widetilde{TM} = TM\backslash\{0\}$, where $a_{ij}(x)$ is a Riemannian metric tensor, were introduced by R.S.Ingarden,[13],[28], and were remarkable studied by M. Matsumoto and his school[41,42,43]. These were suggested by Randers' studies [85] on the geometrical model of the gravitational and electromagnetic fields, a reason to call them "Randers spaces". In addition, R.Miron introduced the notion of general Randers spaces in [50], studied it in detail and applied it in the Relativistic Optics.

On the other hand, Singular Riemannian spaces with the metric tensor field $a_{ij}(x)$ defined on M, where $a_{ij}(x)$ is singular, that is, the rank$(a_{ij}(x))$ is less than the dimension n of the base manifold M, were studied with very interesting results by Gr.Moisil[58] and V.Oproiu[72]. Further, Singular Finsler spaces with the singular fundamental function $F(x,y)$, in which the fundamental tensor field $g_{ij}(x,y) = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^i \partial y^j}$ is singular, were introduced by T.Nagano[60,61,62,63,64,65,66].
Some important problems of singular Finsler spaces have been studied by Prof. A. Bejancu in his book[21].

Therefore, it was necessary to study the following problems:
-A clear definition of singular Finsler space.
-Some good examples, which prove the existence of singular Finsler spaces. Singular Randers spaces, defined by author [62] gives us a natural and remarkable examples.
-Variational problem for singular fundamental function of space SF^n.
-Geodesics.
-Singular metrical connections of these spaces.
-The transformations of singular metrical connections.
-The geometrical methods for studying the singular Finsler spaces.
-The extension of the previous theory to the singular Lagrange spaces.
-The generalized singular Finsler spaces.

But what kind of methods we can used in this study? Of course, the method suggested by Lagrangian study of variational problem is good one. But it is not sufficient for study the metrical connections in singular Finsler space.
In the case of singular metrical connection we must extend the method of Oproiu from the singular Riemannian spaces.
Consequently, we solved the previous problems using new ideas and new methods. Almost all results from the present thesis are original.
Of course the notion of generalized singular Finsler space is new. It is developed here by the methods suggested from singular Finsler spaces.

We develop the contents as each chapter from thesis: We present in this part of the thesis on abstract of the main results obtained in the theory of singular Finsler spaces and in the generalized singular Finsler spaces. Therefore we describe each chapter of thesis.

Chapter 1
Introduction on Finsler, Lagrange and generalized Lagrange spaces.

The theory of singular Finsler spaces or of generalized Finsler spaces is a special case of classical theory of Finsler, Lagrange or generalized Lagrange spaces. Therefore we recall the known theory of Finsler, Lagrange and generalized Lagrange spaces - as preliminaries. This theory was created by my teachers: M. Matumoto, M. Hashiguchi, as well as, in Romania by R. Miron, M. Anastasiei, A. Bejancu, V. Oproiu and many others.

In this chapter, we expose the main results from Ph.D.Thesis. Except the first chapter, almost all thesis contains the original results. Chapter 1 is an abstract of the geometry of Finsler, Lagrange and generalized Lagrange spaces. So we have

Prop.1.2.1 Any Finsler space $F^n = (M, F(x,y))$ determines a Lagrange space $L^n = (M, F^2(x,y))$.

Prop.1.2.2 If the Lagrange space $L^n = (M, L(x,y))$ has the fundamental tensor $g_{ij}(x,y)$ 0−homogeneous with respect to y^i and positive definite, then the pair $F^n = (M, \sqrt{g_{ij}(x,y)y^iy^j})$ is a Finsler space.

In section 1.3, on the canonical nonlinear connection and metrical connection, we remenber:

Theorem 1.3.3 There exists a unique N-connection $L\Gamma(N)$ having the following properties:

$$a.\ g_{ij|k} = 0;\quad b.\ T^i_{jk} = 0;\quad c.\ g_{ij}|_k = 0;\quad d.\ S^i_{jk} = 0.$$

This connection has the local coefficients given by the generalized Christoffel symbols:

$$L^i_{jk} = \frac{1}{2} g^{ir}\left(\frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r}\right),$$

$$C^i_{jk} = \frac{1}{2} g^{ir}\left(\frac{\partial g_{jr}}{\partial y^k} + \frac{\partial g_{kr}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^r}\right).$$

Theorem 1.3.5 If $L^n = (M, L)$ is a Lagrange space, then the differential form

$$\omega = \frac{1}{2}\frac{\partial L}{\partial y^i} dx^i, \qquad \theta = g_{ij}\delta y^i \wedge dx^j$$

are globally defined on $\widetilde{TM}$ and the exterior differential of ω is 2-form θ:

$$d\omega = \theta.$$

§1.4. **Generalized Lagrange spaces.**

Theorem 1.4.1 There exists a unique metrical N-connection $L\Gamma(N) = (L^i_{jk}, C^i_{jk})$ with the properties $T^i_{jk} = 0, S^i_{jk} = 0$. Its coefficients are given by the generalized Christoffel symbols (from Theorem 1.3.3)

Theorem 1.4.4 The set of transformations of N-connections and the composing of mappings is an abelian group isomorphic to the additive group of the pairs of d-tensor fields $(\Omega^{ir}_{sj}X^s_{rk}, \Omega^{ir}_{sj}Y^s_{rk})$.

Theorem 1.4.7 In order that a generalized Lagrange space $GL^n = (M, g_{ij})$ be reducible to a Lagrange space it is necessary that the d-tensor field $\dfrac{\partial g_{ij}}{\partial y^k}$ be totally symmetric.

In the section 4 of chapter 1 we presented an abstract of theory of generalized Lagrange spaces with regular metric.

Theorem 1.5.1(R.Miron[54]) If the generalized Lagrange space GL^n is with weakly regular metric, then the functions $(N^i_j(x,y))$ given by

$$N^i_j(x,y) = \frac{\partial G^i}{\partial y^j}, \qquad G^i = \frac{1}{4} g^{*ij} \left(\frac{\partial^2 \mathcal{E}}{\partial y^j \partial x^k} y^k - \frac{\partial \mathcal{E}}{\partial x^j} \right)$$

determine a nonlinear connection on the total space $\widetilde{TM}$ which depends on the metric tensor $g_{ij}(x,y)$, only.

Theorem 1.5.3 If a generalized Lagrange space GL^n is with regular metric, then the tensor field g_{ij} is 0-homogeneous with respect to (y^i), namely GL^n is a generalized Finsler space.

Theorem 1.5.4(M. Hashiguchi[32]) If a generalized Finsler space GF^n is with regular positively defined metric, then

1) The variational problem for the Lagrangian $L = \mathcal{E}^{1/2}$ is regular.
2) Transversality does not coincide to orthogonality.

Theorem 1.5.6 A Finsler space F^n has the properties:

1) F^n determines a generalized Finsler space $GL^n = (M, g_{ij}(x,y))$ with regular metric.
2) The absolute energy is the square of the fundamental function i.e. $\mathcal{E}(x,y) = F^2(x,y)$.
3) The nonlinear connection (1.42), (1.43) is the Cartan nonlinear connection $\overset{c}{N}$.
4) The canonical connection $C\Gamma(\overset{c}{N})$ is the Cartan metrical connection.

Chapter 2: The notion of Singular Finsler Spaces. Variational Problem.

We continue to present, in abstract, new results from chapter 2.

§2.1 Introduction.

The notion of singular Finsler space was not defined till now. It is very clear in the case of singular Riemannian spaces.

Therefore, in this chapter, we define the concept of singular Finsler space SF^n, as a natural extension of singular Riemannian space. We study the variational problem, the energy and law of conservation of energy of the space SF^n. The theory is applied to study the geodesics of these spaces.

§2.2 Space SF^n.

Let M be a C^∞-real manifold of dimension n and (TM, π, M) its tangent bundle. The canonical coordinate of a point $u \in TM$, are denoted by $(x^i, y^i), (i, j, k, \cdots = 1, 2, \cdots, n)$. We write $u(x^i, y^i)$ or $u = (x,y) \in TM$, where $\pi(u) = x$.

Definition 2.2.1 A singular Finsler space is a pair $SF^n = (M, F(x,y))$, in which F is a mapping from TM to R, $F : (x,y) \in TM \longrightarrow F(x,y) \in R$ satisfying the following axioms

1) F is a differentiable function on $\widetilde{TM} = TM \backslash \{0\}$ and continuous on the null section of the projection π

2) $F(x,y) \geq 0$ on $\widetilde{TM}$

3) F is positively 1-homógeneous with respect to y^i

$$F(x, ty) = tF(x,y), \quad \forall t > 0$$

4) The Hessian of F, with the elements

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \qquad \forall (x,y) \in \widetilde{TM}$$

has the rank $n-k>0$ and the quadratic form $\psi = g_{ij}(x,y)\xi^i\xi^j$ having in canonical form only positively term (namely its canonical form is
$\psi = (\omega^1)^2 + \cdots + (\omega^{n-k})^2$).

The function F is called *fundamental function* of SF^n.

We define the distribution V_1 of nullity of the space SF^n and an apriori fixed complementary distribution V_2 such that the direct sum $V = V_1 \oplus V_2$ holds.

Let $\underset{1}{v}$ and $\underset{2}{v}$ the supplementary projectors with respect to the distributions V_1 and V_2. We denoted by $\underset{1}{v}{}^i_j = l^i_j$, $\underset{2}{v}{}^i_j = m^i_j$ the components of v_1 and v_2.

Prop.2.2.2 The fundamental tensor field g_{ij} and the projectors l^i_j, m^i_j satisfy the following equations

$$g_{ij} l^j_h = 0$$

$$g_{ij} m^j_h = g_{ih}$$

Theorem 2.2.1 With respect to the direct decomposition (1.7), there exist a unique d-tensor field g^{ij} with the properties

$$\begin{cases} g_{ih}g^{hj} = m^j_i \\ g^{ih}\eta^a_h = 0 \end{cases}$$

The tensor g^{ij} is called the generalized inverse of the fundamental tensor field g_{ij}.

Theorem 2.2.2 The distribution of nullity V_1 is integrable if and only if we have

$$\underset{(1)}{R}{}^{\alpha}_{bc} = 0, \qquad (\alpha = k+1, \cdots, n;\ a, b = 1, \cdots, k)$$

Theorem 2.2.3 The distribution V_2 is integrable if and only if we have

$$\underset{(2)}{R}{}^{a}_{\beta\gamma} = 0, \qquad (a = 1, \cdots, k;\ \beta, \gamma = k+1, \cdots, n).$$

Here

$$\underset{(1)}{R}{}^{c}_{ab}, \underset{(1)}{R}{}^{\alpha}_{ab}, \underset{(2)}{R}{}^{a}_{\beta\gamma}, \underset{(2)}{R}{}^{\alpha}_{\beta\gamma}, \underset{(1)}{B}{}^{a}_{b\beta}, \underset{(2)}{B}{}^{\alpha}_{b\beta}$$

are the invariants of the space SF^n.
We remark that the invariants $\underset{(1)}{R}{}^{c}_{ab}$ does not depend on the distribution V_2.

§2.3 **Variational problems.**

Theorem 2.3.1 In order that the functional $I(c)$ be an extremal value of $I(c_\varepsilon)$ it is necessary that the curve c be solution of the Euler-Lagrange equations

$$E_i(F^2) := \frac{\partial F^2}{\partial x^i} - \frac{d}{dt}\frac{\partial F^2}{\partial y^i} = 0, \qquad y^i = \frac{dx^i}{dt}.$$

Theorem 2.3.2 The following properties hold:

1) $E_i(F^2)$ is a d-covector field.

2) $E_i(F^2 + F'^2) = E_i(F^2) + E_i(F'^2), \quad E_i(aF^2) = aE_i(F^2),\ a \in R.$

3) $E_i\left(\frac{df}{dt}\right) = 0,\ \forall f \in \mathfrak{F}(TM)$, with $\frac{\partial f}{\partial y^i} = 0.$

Prop.2.3.1 The Hamiltonian energy (2.9) of the space SF^n is $E_{F^2} = F^2$

Theorem 2.3.3 The energy function F^2 of the singular Finsler space $SF^n = (M, F)$ is constant along the every integral curve of the Euler-Lagrange equation $E_i(F^2) = 0$.

Theorem 2.3.4 For a singular Finsler space $SF^n = (M, F)$, the Euler-Lagrange equation has the form

$$g_{ij}\frac{d^2x^j}{dt^2} + [jk, i]\frac{dx^j}{dt}\frac{dx^k}{dt} = 0,$$

where $[jk, i]$ are the Christoffel symbols of the first type, of $g_{ij}(x, y)$.

Prop.2.3.2 The projections of the covector field $E_i(F^2)$ on the distributions V_2 and V_1 are given, respectively by

$$\begin{cases} m^i_j E_i(F^2) = -2\left(g_{jr}\frac{d^2x^r}{dt^2} + m^i_j[rs, i]\frac{dx^r}{dt}\frac{dx^s}{dt}\right) & \text{on } V_2 \\ l^i_j E_i(F^2) = -2\left(l^i_j[rs, i]\frac{dx^r}{dt}\frac{dx^s}{st}\right) & \text{on } V_1. \end{cases}$$

Theorem 2.3.5 The Euler-Lagrange equation $E_i(F^2) = 0$ holds if and only if

$$\begin{cases} g_{ji}\left(\frac{d^2x^i}{dt^2} + \{{}_r{}^i{}_s\}\frac{dx^r}{dt}\frac{dx^s}{dt}\right) = 0 \\ l^i_j[rs, i]\frac{dx^r}{dt}\frac{dx^s}{st} = 0, \end{cases}$$

where we denote

$$\{{}_j{}^i{}_k\} = g^{is}[jk, s].$$

§2.4 **Geodesics.**

Theorem 2.4.1 The equation of geodesics in the natural parameterization are given by equation (3.5).

Theorem 2.4.2 The equations (3.1) of the geodesics of singular Finsler space $SF^n = (M, F(x,y))$ are equivalent to the following system of differential equations

$$\begin{cases} g_{ji}\Big(\dfrac{d^2x^i}{ds^2} + \{{}_r{}^i{}_m\}\dfrac{dx^r}{ds}\dfrac{dx^m}{ds}\Big) = 0, \\ l^i_j[rm,i]\dfrac{dx^r}{ds}\dfrac{dx^m}{ds} = 0. \end{cases}$$

Theorem 2.4.3 The geodesic covector field of a geodesic curve of singular Finsler space SF^n belongs to the distribution V_2.

An example:

The pair $SF^n = (M, \sqrt{a_{ij}(x)y^iy^j})$ is a singular Finsler space, where $a_{ij}(x)$ is a singular Riemannian metric.

Theorem 2.5.2 A singular Finsler space $SF^n = (M, F(x,y))$ is reducible to a singular Riemannian space if and only if the d-tensor field

$$C_{ijk} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}$$

vanishes.

Theorem 2.5.3 If the manifold M is endowed with a singular Riemannian metric, then locally on M there exist singular Finsler spaces $SF^n = (M, F(x,y))$.

Chapter 3
Singular Randers Spaces.

In Chapter 3 we study a first important class of spaces SF^n. Namely, we give :

Definition 3.2.2 A singular Randers space is a singular Finsler space with the fundamental function $F(x,y)$ from:

$$F(x,y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i,$$

defined in every point $u \in \pi^{-1}(U)$ and where $a_{ij}(x)$ is a singular Riemannian metric.

Theorem 3.1.1 With respect to the direct decomposition (3.6), there exist a unique $d-$tensor field $a^{ij}(x)$ with the properties

$$\begin{cases} a_{ih}(x)a^{hj}(x) = m^j_i(x) \\ a^{ih}(x)\eta^a_h(x) = 0 \end{cases}$$

Theorem 3.2.1 1) If $a_{ij} - b_i b_j$ is positive semi-definite, then the fundamental function $F(x,y)$ defined by (3.15) is semi-positive valued ($F \geq 0$) on the domain D and differentiable on $\widetilde{D}$, where

$$D = \{(x,y) \in \widetilde{TM} | \alpha \geq 0\} \quad \text{and} \quad \widetilde{D} = \{(x,y) \in \widetilde{TM} | \alpha > 0\} \subset D.$$

2) The metric tensor $g_{ij}(x,y)$ of the singular Randers space SF^n is given by (3.20).
3) $g_{ij}(x,y)$ has the same rank with the singular Riemannian metric $a_{ij}(x)$, namely

$$rank(g_{ij}(x,y)) = rank(a_{ij}(x)).$$

We construct two examples of Randers spaces. The first one is as follows:
Example 1. Let E^2 be the Euclidean plan with an orthonormal coordinate system (x,y). At an arbitrary point $P(x,y)$ of $\widetilde{E^2} = E^2 - \{0\}$ we define the indicatrix curve I_P(Figure 1) such that I_P is parallel lines with the line and the distance to the line Oy is $\sqrt{1+e(x,y)}|x|$, where $e(x,y)$ is a positive-valued function. Then I_P is given by the equation

$$u^2 = (1+e(x,y))x^2$$

in the coordinates (u,v). Since the tangent space of E^2 at P can be identified with E^2 itself, we may put $u = x + \dot{x},\ v = y + \dot{y}$. Then (3.32) is written as

$$e(x,y)x^2 - 2x\dot{x} - \dot{x}^2 = 0.$$

Now we apply the usual method to (3.33). Replacing $(\dot{x},\dot{y})$ by $(\frac{\dot{x}}{F}, \frac{\dot{y}}{F})$ we get

$$e(x,y)x^2F^2 - 2x\dot{x}F - \dot{x}^2 = 0.$$

This algebraic equation for F has two solutions, one is positive and the other negative. We choose the positive solution

$$F = \frac{1}{e(x,y)}\left(\sqrt{1+e(x,y)}\left|\frac{\dot{x}}{x}\right| + \frac{\dot{x}}{x}\right).$$

Thus we obtain a two-dimensional Singular Randers space $(\widetilde{E^2}, F)$, where the singular Randers metric F is given by (3.35).

As the particular cases we have

$$e = x^{2p} + y^{2p},\ p \geq 1, \qquad \forall (x,y) \in \widetilde{E^2}$$

and we obtain interesting examples of Randers spaces.

§3.3 Variational problem.

Theorem 3.3.1 The necessary conditions for the nonisotropic curve $c : s \longrightarrow (x^i(s))$ to be an extremal value of (3.26)is that the functions $x^i(s)$ be a solution of the following differential equations

$$a_{ir}\frac{d^2x^r}{ds^2} + [rk,i]\frac{dx^r}{ds}\frac{dx^k}{ds} = F_{ir}\frac{dx^r}{ds},$$

where s satisfies (3.24), that is , $ds = \alpha(x,dx)$ and $[rk,i] = \frac{1}{2}\left(\frac{\partial a_{ir}}{\partial x^k} + \frac{\partial a_{ik}}{\partial x^r} - \frac{\partial a_{rk}}{\partial x^i}\right)$.

Theorem 3.3.2 1) The equations (3.29) of the geodesics of singular Randers spaces $SF^n = (M, \alpha + \beta)$ are equivalent to the system of differential equations (3.31)
2) The geodesics of the space SF^n belongs in the distribution of nullity V_1 if the second condition of

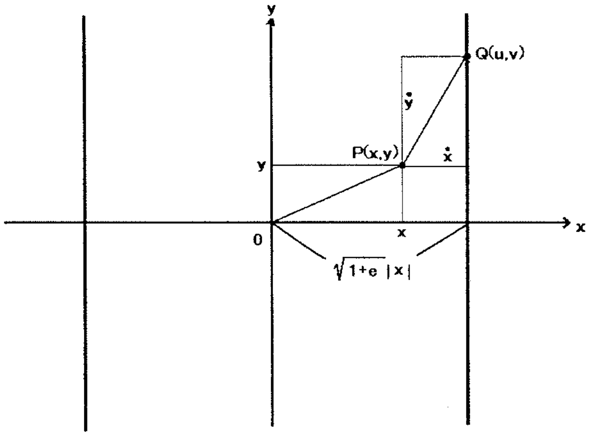

Figure 1: Example 1

(3.31) is identically satisfied.

Chapter 4
Variational Problems in the Singular Lagrange Spaces.

In Chapter 4 we study the notion of singular Lagrange space following the same methods as in the case of spaces SF^n.

The section 2 from this chapter is devoted to the variational problem and Euler-Lagrange equations.

Theorem 4.2.1 In order that the functional $I\,(c)$ be an extremal value of $I(c_\epsilon)$ it is necessary that c be the solution of the Euler-Lagrange equations

$$E_i(\mathcal{L}) \stackrel{def}{=} \frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y^i}\right) = 0, \quad y^i = \frac{dx^i}{dt}.$$

Theorem 4.2.2 The following properties hold:
1) $E_i(\mathcal{L})$ is a d-covector field.
2) $E_i(\mathcal{L}+\mathcal{L}') = E_i(\mathcal{L}) + E_i(\mathcal{L}')$, $E_i(a\mathcal{L}) = aE_i(\mathcal{L})$, $a \in R$
3) $E_i(\frac{df}{dt}) = 0$, $\forall f \in \mathcal{F}(TM)$, with $\frac{df}{dy^i} = 0$

Theorem 4.2.3 The energy $E_{\mathcal{L}}$ of the singular Lagrangian $\mathcal{L}$ is conserved along to every integral curve c of the Euler-Lagrange equation $E_i = 0$, $y^i = \frac{dx^i}{dt}$.

Theorem 4.2.4 For a singular Lagrange space $S\mathcal{L}^n$, the Euler-Lagrange equation has the form

$$2g_{ir}\frac{d^2x^r}{dt^2} + \frac{\partial^2 \mathcal{L}}{\partial x^r \partial y^i}y^r - \frac{\partial \mathcal{L}}{\partial x^i} = 0, \quad y^i = \frac{dx^i}{dt}$$

Prop.4.2.1 The projection of the covector field $E_i(\mathcal{L})$ on the distributions V_2 and V_1 are given, respectively by

$$\begin{cases} m^i_j E_i(\mathcal{L}) = -2\left(g_{jr}\frac{d^2x^r}{dt^2} + \frac{1}{2}m^i_j\left(\frac{\partial^2\mathcal{L}}{\partial x^r\partial y^i}y^r - \frac{\partial\mathcal{L}}{\partial x^i}\right)\right) & on \quad V_2 \\ l^i_j E_i(\mathcal{L}) = -l^i_j\left(\frac{\partial^2\mathcal{L}}{\partial x^r\partial y^i}y^r - \frac{\partial\mathcal{L}}{\partial x^i}\right) \quad on \quad V_1 \end{cases}$$

Theorem 4.2.5 The Euler - Lagrange equation $E_i(L) = 0$ holds if and only if

$$\begin{cases} g_{jr}\left(\frac{d^2x^r}{dt^2} + 2G^r(x,y)\right) &= 0 \\ l^i_j\left(\frac{\partial^2\mathcal{L}}{\partial x^r\partial y^i}y^r - \frac{\partial\mathcal{L}}{\partial x^i}\right) &= 0 \end{cases}$$

where

$$G^r(x,y) = \frac{1}{4}g^{ri}\left(\frac{\partial^2\mathcal{L}}{\partial x^s\partial y^i}y^s - \frac{\partial\mathcal{L}}{\partial x^i}\right), \ y^r = \frac{dx^r}{dt}$$

Definition 4.1.3 A singular Lagrange space whose fundamental function is given by

$$\mathcal{L} = F^2(x,y) + A_i(x)y^i + U(x),$$

where F is a fundamental function of a singular Finsler space $(M, F(x,y))$, will be called a *Caratheodory* singular Finslerian Lagrange space, shortly $SAF\mathcal{L}-$ space.

Theorem 4.2.6 For a *Caratheodory* $SAF\mathcal{L}$ - space, the Euler- Lagrange equation has the form

$$g_{ir}\frac{d^2x^r}{dt^2} + [rs,i]\frac{dx^r}{dt}\frac{dx^s}{dt} = F_{ij}(x)\frac{dx^j}{dt}$$

where $[rs,i]$ are the Christoffel symbols of the first type and

$$F_{ij}(x) = \frac{1}{2}\left(\frac{\partial A_i(x)}{\partial x^j} - \frac{\partial A_j(x)}{\partial x^i}\right)$$

is electromagnetic tensor field.

Prop.4.2.2 The projection of the covector field $E_i(\mathcal{L})$ on the distributions V_2 and V_1 are given, respectively, by

$$\begin{cases} m^i_j E_i(\mathcal{L}) = -2\left(g_{jk}\frac{d^2x^k}{dt^2} + m^i_j[rs,i]\frac{dx^r}{dt}\frac{dx^s}{dt} - m^i_j F_{ir}\frac{dx^r}{dt}\right) & on \quad V_2 \\ l^i_j E_i(\mathcal{L}) = -2\left(l^i_j[rs,i]\frac{dx^r}{dt}\frac{dx^s}{dt} - l^i_j F_{ir}\frac{dx^r}{dt}\right) \quad on \quad V_1 \end{cases}$$

Theorem 4.2.7 The Euler - Lagrange equation $E_i(\mathcal{L}) = 0$ holds if and only if

$$\begin{cases} g_{jk}\left(\frac{d^2x^k}{dt^2} + \{{}_r{}^k{}_s\}\frac{dx^r}{dt}\frac{dx^s}{dt} - F^k_h\frac{dx^h}{dt}\right) = 0 \\ l^i_j\left([rs,i]\frac{dx^r}{dt}\frac{dx^s}{dt} - F_{ir}\frac{dx^r}{dt}\right) = 0 \end{cases}$$

where $\{{}_r{}^k{}_s\} = g^{ki}[rs,i]$ and $g^{ki}F_{ih} \overset{not}{=} F^k_h$.

Theorem 4.2.8 The Euler - Lagrange equation $E_i(\mathcal{L}) = 0$ of a $SAF\mathcal{L} - space$ holds if and only if

$$\begin{cases} \gamma_{jk}(x)\left(\frac{d^2x^k}{dt^2} + \{{}_r{}^k{}_s\}\frac{dx^r}{dt}\frac{dx^s}{dt} - F^k_h\frac{dx^h}{dt}\right) = 0 \\ l^i_j\left([rs,i]\frac{dx^r}{dt}\frac{dx^s}{dt} - F_{ir}\frac{dx^r}{dt}\right) = 0 \end{cases}$$

where $\{{}_{r}{}^{j}{}_{s}\} = g^{ji}[rs, i]$.

Chapter 5
On the Connections of Singular Finsler spaces.

The Chapter 5 is devoted to the N-linear connections which are singular with respect to $g_{ij}(x, y)$ and metrical singular with respect to $g_{ij}(x, y)$
In the whole chapter 5 we systematically used the Oproiu's method and Obata-Oproiu operators.

For singular Finsler connections, we have some new important results. A singular Finsler connection is an N-linear connection $D\Gamma(N)$ with the properties $g_{ir|k}g^{rj} = 0$ and $g_{ir}\overset{\cdot}{|}_{k}g^{rj} = 0$. So we have:

Prop.5.2.1 The equations (5.41) and (5.42) are equivalent to respectively, the equations:

$$(\Psi + \Theta)^{si}_{jr} D^{r}_{ks} = \frac{1}{2} g_{js}\overset{\cdot}{|}_{k} g^{si} + \frac{1}{2} g_{rs}\overset{\cdot}{|}_{k} l^{r}_{j} g^{si},$$
$$(\Psi + \Theta)^{si}_{jr} E^{r}_{ks} = \frac{1}{2} g_{js|k} g^{si} + \frac{1}{2} g_{rs|k} l^{r}_{j} g^{si}.$$

The operators $\underset{1}{\Omega} = \Psi - \Theta$, $\underset{2}{\Omega} = \Psi + \Theta$ are the Obata-Oproiu's operators.

Prop.5.2.2 If we put

$$\bar{D}^{i}_{kj} = \frac{1}{2} g_{js}\overset{\cdot}{|}_{k} g^{si} + \frac{1}{2} g_{rs}\overset{\cdot}{|}_{k} l^{r}_{j} g^{si},$$
$$\bar{E}^{i}_{kj} = \frac{1}{2} g_{js|k} g^{si} + \frac{1}{2} g_{rs|k} l^{r}_{j} g^{si},$$

then the general solutions of (5.44) and (5.45) are

$$D^{l}_{hk} = \bar{D}^{l}_{hk} + (\Phi - \Theta)^{jl}_{ki} P^{i}_{hj},$$
$$E^{l}_{hk} = \bar{E}^{l}_{hk} + (\Phi - \Theta)^{jl}_{ki} Q^{i}_{hj},$$

where P^{i}_{kj}, Q^{i}_{kj} are arbitrary d-tensor fields.

Theorem 5.2.1 Let $F\dot{\Gamma} = (\dot{N}^{i}_{j}, \dot{F}^{i}_{j}{}_{k}, \dot{C}^{i}_{j}{}_{k})$ be a fixed Finsler connection. Then the set of all singular Finsler connections $F\Gamma = (N^{i}_{j}, F^{i}_{j}{}_{k}, C^{i}_{j}{}_{k})$ is given by

1) $N^{i}_{j} = \dot{N}^{i}_{j} + A^{i}_{j}$,

2) $F^{i}_{j}{}_{k} = \dot{F}^{i}_{j}{}_{k} - (\dot{C}^{i}_{r}{}_{k} + D^{i}_{r}{}_{k}) A^{r}_{j} + E^{i}_{j}{}_{k}$,

3) $C^{i}_{j}{}_{k} = \dot{C}^{i}_{j}{}_{k} + D^{i}_{j}{}_{k}$,

where

$D^{l}_{hk} = \frac{1}{2} g_{ks}\overset{\cdot}{|}_{h} g^{sl} + \frac{1}{2} g_{rs}\overset{\cdot}{|}_{h} l^{r}_{k} g^{sl} + (\Phi - \Theta)^{jl}_{ki} P^{i}_{hj}$,

$E^{l}_{hk} = \frac{1}{2} g_{ks|h} g^{sl} + \frac{1}{2} g_{rs|h} l^{r}_{k} g^{sl} + (\Phi - \Theta)^{jl}_{ki} Q^{i}_{hj}$

and $A^{i}_{j}, P^{i}_{j}{}_{k}, Q^{i}_{jk}$ are arbitrary d-tensor fields and $\Phi^{jl}_{hi}, \Theta^{jl}_{hi}$ are the quantities of (5.40).

Theorem 5.2.2 Let $F\dot{\Gamma} = (\dot{N}^{i}_{j}, \dot{F}^{i}_{j}{}_{k}, \dot{C}^{i}_{j}{}_{k})$ be a fixed Finsler connection. Then the following Finsler connection $(N^i_j, F^i_{jk}, C^i_{jk})$

$$\begin{aligned} N^i_j &= \dot{N}^i_j, \\ F^i_{jk} &= \dot{F}^i_{jk} + \frac{1}{2} g_{ks|j} g^{si} + \frac{1}{2} g_{rs|j} l^r_k g^{si}, \\ C^i_{jk} &= \dot{C}^i_{jk} + \frac{1}{2} g_{ks}\dot{|}_j g^{si} + \frac{1}{2} g_{rs}\dot{|}_j l^r_k g^{si} \end{aligned}$$

is a singular Finsler connection.

This result proves the existence of singular Finsler connections.

Theorem 5.2.3 The set of all singular Finsler connections $(N^i_j, F^i_{jk}, C^i_{jk})$ is given by

$$\begin{aligned} N^i_j &= \dot{N}^i_j + A^i_j, \\ F^i_{jk} &= \dot{F}^i_{jk} - \dot{C}^i_{rk} A^r_j + (\Phi - \Theta)^{hi}_{kr} (Q^r_{jh} - P^r_{lh} A^l_j), \\ C^i_{jk} &= \dot{C}^i_{jk} + (\Phi - \Theta)^{hi}_{kr} P^r_{jh} \end{aligned}$$

where $F\dot{\Gamma} = (\dot{N}^i_j, \dot{F}^i_{jk}, \dot{C}^i_{jk})$ is a fixed singular Finsler connection and $A^i_j, P^i_{jk}, Q^i_{jk}$ are arbitrary d-tensor fields.

Theorem 5.2.5 The set of all singular Finsler connections $F\Gamma(\dot{N})$ is given by

$$\begin{aligned} N^i_j &= \dot{N}^i_j, \\ F^i_{jk} &= \dot{F}^i_{jk} + (\Phi - \Theta)^{hi}_{kr} Q^r_{jh}, \\ C^i_{jk} &= \dot{C}^i_{jk} + (\Phi - \Theta)^{hi}_{kr} P^r_{jh} \end{aligned}$$

where $F\dot{\Gamma}(\dot{N})$ is a fixed singular Finsler connection and P^i_{jk}, Q^i_{jk} are arbitrary d-tensor fields.

Theorem 5.2.6 The set of transformations of singular Finsler connections and the composition of mappings is an abelian group, isomorphic to the additive group of pairs of tensors:

$$\{(\Phi - \Theta)^{hi}_{kr} Q^r_{jh}, (\Phi - \Theta)^{hi}_{kr} P^r_{jh}\}.$$

§5.3 is devoted to:
The metrical property of singular Finsler connections. A metrical singular Finsler connection is defined by the equations $g_{ij|k} = 0$ and $g_{ij}\dot{|}_k = 0$. So, any metrical singular Finsler connection is a singular Finsler connection.

Theorem 5.3.1 The set of all metrical singular Finsler connections is given by $F\Gamma = (N^i_j, F^i_{jk}, C^i_{jk})$, where

$$\begin{aligned} N^i_j &= \dot{N}^i_j + A^i_j, \\ F^i_{jk} &= \dot{F}^i_{jk} - \dot{C}^i_{rk} A^r_j + (\Phi - \Theta)^{hi}_{kr} (Q^r_{jh} - P^r_{lh} A^l_j), \\ C^i_{jk} &= \dot{C}^i_{jk} + (\Phi - \Theta)^{hi}_{kr} P^r_{jh} \end{aligned}$$

where $F\dot{\Gamma} = (\dot{N}^i_j, \dot{F}^i_{jk}, \dot{C}^i_{jk})$ is a fixed Finsler connection and $A^i_j, P^i_{jk}, Q^i_{jk}$ are arbitrary d-tensor fields.

Theorem 5.3.2 Let $F\dot{\Gamma} = (\dot{N}^i_j, \dot{F}^i_{jk}, \dot{C}^i_{jk})$ and A^i_j, P^i_{jk} be a fixed metrical singular Finsler connection and arbitrary d-tensor fields. Then the following Finsler connection $(N^i_j, F^i_{jk}, C^i_{jk})$:

$$\begin{aligned} N^i_j &= \dot{N}^i_j + A^i_j, \\ F^i_{jk} &= \dot{F}^i_{jk} - \dot{C}^i_{rk} A^r_j \\ C^i_{jk} &= \dot{C}^i_{jk} + (\Phi - \Theta)^{hi}_{kr} P^r_{jh} \end{aligned}$$

is a metrical singular Finsler one.

Theorem 5.3.3 The set of all metrical singular Finsler connections $F\Gamma(\dot{N})$ is given by

$$\begin{aligned} N^i_j &= \dot{N}^i_j, \\ F^i_{j\,k} &= \dot{F}^i_{j\,k} + (\Phi - \Theta)^{hi}_{kr} Q^r_{j\,h}, \\ C^i_{j\,k} &= \dot{C}^i_{j\,k} + (\Phi - \Theta)^{hi}_{kr} P^r_{j\,h} \end{aligned}$$

where $F\dot{\Gamma}(\dot{N})$ is a fixed metrical singular Finsler connection and $P^i_{j\,k}, Q^i_{j\,k}$ are arbitrary *d*-tensor fields.

§5.4 The torsion tensor field of the metrical singular Finsler connections.
We look for *d*-tensor of torsion of the metrical singular Finsler connections which do not depend on the distribution V_2. Theorem 5.4.1 solved this problem.

Prop. 5.4.1 For the d-tensor fields Q^i_{jk}, P^i_{jk} in the formulas (5.72) and (5.73), the following equations

$$\begin{aligned} \frac{1}{2}(m^i_l m^j_k - g_{lk} g^{ij}) Q^k_{ri} m^r_s &= \frac{1}{2} m^r_l g_{iq|r} m^i_s g^{qj} - \frac{1}{2} g_{ir|q} m^i_s m^r_l g^{qj}, \\ \frac{1}{2}(m^i_l m^j_k - g_{lk} g^{ij}) P^k_{ri} m^r_s &= \frac{1}{2} m^r_l g_{iq}|_r m^i_s g^{qj} - \frac{1}{2} g_{ir}|_q m^i_s m^r_l g^{qj} \end{aligned}$$

are satisfied, respectively.

Theorem 5.4.1 Let $F\dot{\Gamma} = (\dot{N}^i_j, \dot{F}^i_{j\,k}, \dot{C}^i_{j\,k})$ be a fixed Finsler connection satisfying $\dot{T}^i_{j\,k} = 0$ and $\dot{S}^i_{j\,k} = 0$, and let the metrical singular Finsler connection $F\Gamma = (\dot{N}^i_j, F^i_{j\,k}, C^i_{j\,k})$ be given by

$$\begin{aligned} F^i_{j\,k} &= \dot{F}^i_{j\,k} + E^i_{j\,k}, \\ C^i_{j\,k} &= \dot{C}^i_{j\,k} + D^i_{j\,k}. \end{aligned}$$

The torsion tensor fields $T^i_{jk}, S^i_{jk}$of $F\Gamma(\dot{N})$ have the properties $m^i_r T^r_{kj} m^k_s m^j_l = 0$ and $m^i_r S^r_{kj} m^k_s m^j_l = 0$, respectively if and only if d-tensor fields E^i_{hj}, D^i_{hj} satisfy the property (5.81) and (5.82) with Q^i_{jk}, P^i_{jk} satisfying (5.79) and (5.80), respectively.

Prop.5.4.2 The torsion tensor fields T^i_{jk}, S^i_{jk} are written in the form:

$$\begin{aligned} T^i_{jk} = \frac{1}{2} L_{krs} m^r_j g^{si} - \frac{1}{2} L_{jrs} m^r_k g^{si} + L_{krs} l^r_j g^{si} + \\ +(\Phi - \Theta)^{li}_{kt} (\frac{1}{2} l^t_r T^r_{jl} + m^t_r T^r_{sl} l^s_j) - (\Phi - \Theta)^{li}_{jt} (\frac{1}{2} l^t_r T^r_{kl} + m^t_r T^r_{sl} l^s_k) \end{aligned}$$

and

$$\begin{aligned} S^i_{jk} = \frac{1}{2} W_{krs} m^r_j g^{si} - \frac{1}{2} W_{jrs} m^r_k g^{si} + W_{krs} l^r_j g^{si} + \\ +(\Phi - \Theta)^{li}_{kt} (\frac{1}{2} l^t_r W^r_{jl} + m^t_r W^r_{sl} l^s_j) - (\Phi - \Theta)^{li}_{jt} (\frac{1}{2} l^t_r W^r_{kl} + m^t_r W^r_{sl} l^s_k). \end{aligned}$$

Theorem 5.4.2 The torsion tensor fields $\bar{T}^i_{jk}, \bar{S}^i_{jk}$ of the metrical singular Finsler connection $F\bar{\Gamma} = (\bar{N}^i_j, \bar{F}^i_{jk}, \bar{C}^i_{jk})$ determined by (5.98) and (5.99) vanish if and only if $L_{ijk} = 0, W_{ijk} = 0$ are satisfied.

Chapter 6
Generalized Singular Finsler Spaces.

In the last chapter we investigated a new notion of the singular Finsler metric. This metric is a generalized Finsler metric. It is defined in the first section of the chapter 6. So, one introduced the notion of generalized singular Finsler spaces. It is a pair $GSF^n = (M, g_{ij}(x,y))$ where g_{ij} is a d-tensor field homogeneous of degree 0 symmetric and of $rank(g_{ij}) = n - k$.

The geometry of GSF^n-spaces can be developed by the same method as the geometry of singular Finsler spaces SF^n.

But the geometry of GSF^n-spaces can not be reduced to the geometry to the singular Finsler spaces or singular Lagrange spaces because the fundamental tensor g_{ij} of the space GSF^n, in general, is not a fundamental tensor of the space SF^n or SL^n. In this case we must study new notions by means the absolute energy of space. We proved that the absolute energy of space GSF^n is coincident with the energy of space (Proposition 6.3.1) and Euler-Lagrange equation are given in the Theorem 6.3.1. Also we study the distribution of nullity of tensor g_{ij} and determined the inverse generalized g^{ij} of g_{ij}.

We established the invariants R^a_{bc} which does not depend on the distribution V_2. We study a co-variant derivation operator which preserve by the parallelism of the distribution N, V_1, V_2. So we study the coefficients of N-linear connection in adopted frames (6.43). In order to establish the geometrical object fields of the space GSF^n which do not depend on the distribution V_2 we investigate the N-linear connections D which preserve by parallelism the distributions N, V_1, V_2. The corresponding $h-, v_1-, v_2-$ covariant derivatives are pointed out. The torsion of curvature of D is studied, too. In finally section, we determined the existence an arbitrary of the set of all metrical singular connections of the generalized singular Finsler spaces.

With these problems we end the text of Ph.D thesis.

References are given only by papers which have some connections with the problems from this Ph.D thesis.

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