Is the Friedmann Universe Machian?

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Abstract

The dragging effect on the inertial frame is investigated in the Friedmann universe. The dragging coefficient ω_0/ω_s decreases as the universe expands. The inertial frame is not dominated completely by matter of the universe. The Friedmann universe is not Machian.

I. Introduction

It is well known that Thirring effect[1),2),3)] by a rotating shell appears in general relativity, and that the factor GM/Rc^2 of the shell plays an important role. In fact the force induced in the vicinity of the orgin of the rotating shell behaves like the Coriolis force when the factor GM/Rc^2 is equal to 3/4.

The motivation of a series of investigations by the author originates in a simple question: What happens if that rotating shell expands ? As the shell expands, the factor GM/Rc^2 decreases; even in this situation, does the Coriolis force remain ? In other words, is the inertial frame at the shell center dominated by the rotating shell itself ? This is the most immediate test to check the validity of Mach's principle in a native sence, which means that the inertial frame is determined completely by matter of the universe.

However, a shell in empty space is a shell to the last, and not the universe. We consider first the homogeneous and isotropic distribution of matter and discuss the inertial frame dragging induced by a spherical rotating shell which has the same density as matter of the universe in a similar method applied by Lausberg[4] to the static *Einstein universe*.

It is natural that we have examined the dragging effect in the closed expanding Friedmann universe. We, however, have found some difficulties, which concludes that the Friedmann universe is not Machian, and which suggests that the gravitational constant should be time-varying in the Machian point of view.

In line with this suggestion we have surveyed a new cosmological solution[5),6)] in the Brans-Dicke theory of gravitation[7)] in succes. In this closed cosmological model the factor $G(t)M/c^2a(t)$ keeps constant owing to the variable gravitational "constant" notwithstanding the universe expands, and moreover the value of that factor is inevitably fixed to π by the theory. The dragging coefficient between the angular velocity of the inertial frame at the origin of the universe and that of the rotating shell reaches unity when the shell covers the whole universe[8)], and hence this cosmological model is Machian in the above-mentioned meaning.

We, in this paper, discuss the inertial frame dragging in the Friedmann universe in the framework of general relativity. Discussions are developed in parallel with the previous investigation[8)] in the closed cosmological model of the Brans-Dicke theory to make clear correspondence and differences between them.

II. The Dragging Effect on the Inertial Frame

We take three angular coordinates $x^1=\chi$, $x^2=\theta$, $x^3=\varphi$ as spatial coordinates and the variable t as the cosmological time, and then the closed Friedmann universe is described by the following metric tensor:

(1) $ds^2=-dt^2+a^2(t)[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta\, d\varphi^2)]$.

The expansion parameter a(t) obeys the Friedmann equation:

(2) $\dot{a}^2=\frac{\kappa Mc^2}{6\pi^2 a}+\frac{1}{3}\lambda a^2-1$,

where κ and λ are the Einstein gravitational constant and the cosmological constant respectively, and M is a integral constant, which means the mass of the whole universe

(3) $M=2\pi^2 a^3\rho$.

Let us consider in this universe a shell, the volume of which is restricted by the two hypersurfaces $\chi=\chi_0$ and $\chi=\chi_1$, with $0<\chi_0<\chi_1<\pi$. The density of the shell is assumed to be the same as the remaining part of the universe. This shell is now considered to be slowly rotating as a rigid body around the axis $\theta=0$, with an angular veocity ω_s relative to the remaining part of the universe.

The metric form in the whole universe will be perturbed by rotation of the shell as

(4) $ds^2=-dt^2+a^2(t)\{d\chi^2+\sin^2\chi[d\theta^2+\sin^2\theta(d\varphi-\omega dt/c)^2]\}$.

Owing to the slow rate of rotation we may limit the calculations up to the first order of an angular velocity ω, that is, to the Coriolis force.

Now we start the following perturbed metric tensor:

(5) $ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\varphi^2)]$
$\quad -2\omega(\chi,\theta,t)a^2(t)\sin^2\chi\sin^2\theta\, d\varphi dt/c.$

Our problem is to find solutions of unknown functions a(t) and $\omega(\chi, \theta, t)$, which obey the Einstein field equations of gravitation:

(6) $R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} - \lambda g_{\mu\nu} = \kappa T_{\mu\nu},$

where $T_{\mu\nu}$ is the energy-momentun tensor, which has the following form for the perfect fluid:

(7) $T_{\mu\nu} = -pg_{\mu\nu} - (\rho + p/c^2)u_\mu u_\nu.$

In the present problem, as the pressure p is negligible, the non-vanishing components of energy-momentum tensor are

(8) $\begin{cases} T_{00} = -\rho c^2 \\ T_{30} = T_{03} = -\rho c(\omega - \omega_s)a^2\sin^2\chi\sin^2\theta \end{cases}$

($\omega_s = 0$ outside the shell).

In the long run (see Appendix A) the independent field equations of gravitation are

(9a) $2a\ddot{a} + \dot{a}^2 - \lambda a^2 + 1 = 0,$

(9b) $-3(\dot{a}^2 + 1)/a^2 + \lambda = -\kappa\rho c^2,$

(9c) $\partial_\chi\dot{\omega} + 3(\dot{a}/a)\partial_\chi\omega = 0$

(9d) $\partial_\theta\dot{\omega} + 3(\dot{a}/a)\partial_\theta\omega = 0$

(9e) $2\omega\sin^2\chi(2a\ddot{a} - \lambda a^2 + 3) + \sin^2\chi(\partial^2_{\chi\chi}\omega + 4\cot\chi\,\partial_\chi\omega - 4\omega)$
$\quad + (\partial^2_{\theta\theta}\omega + 3\cot\theta\,\partial_\theta\omega) = 2\kappa\rho c^2(\omega - \omega_s)a^2\sin^2\chi$

($\omega_s = 0$ outside the shell).

Equations (9a) and (9b) are the same as the unperturbed, and determine completely the time dependence of a(t) and $\rho(t)$, that is, Eqs. (2) and (3). By integrating Eqs. (9c) and (9d) we have

(10) $\omega(\chi,\theta,t)=W(\chi,\theta)/a^3(t)+T(t)$,

where an arbitrary function W depends only on χ and θ, and a function T on t. By substituting Eq. (10) into Eq. (9e) we find that T(t) must be zero substantially in the homogeneous equation of Eq. (9e). By substituting Eqs. (2), (3), and (9a) into Eq. (9e) we have

$$(12)\quad 2\omega\sin^2\chi\left(-\frac{2\kappa Mc^2}{3\pi^2 a}-\frac{1}{3}\lambda a^2+3\right)$$
$$+\sin^2\chi(\partial^2_{\chi\chi}\omega+4\cot\chi\,\partial_\chi\omega-4\omega)+(\partial^2_{\theta\theta}\omega+3\cot\theta\,\partial_\theta\omega)$$
$$=-\frac{\kappa Mc^2}{\pi^2 a}\cdot\omega_s\sin^2\chi$$

($\omega_s=0$ outside the hell),

which determines the inertial property of the universe. We find easily that a particular solution of this inhomogeneous equation is

$$(13)\quad \omega_p=\frac{\kappa Mc^2}{2\pi^2 a}\left(\frac{2\kappa Mc^2}{3\pi^2 a}+\frac{1}{3}\lambda a^2-1\right)^{-1}\cdot\omega_s.$$

The homogeneous equation of Eq. (12) admits a variable separation with respect to χ and θ; let us write

(14) $W(\chi,\theta)=X(\chi)\cdot\Theta(\theta)$.

Denoting by S the separation constant, two equations arise from the homogeneous equation of Eq. (12):

$$(15)\quad \frac{1}{\Theta}\left(\frac{d^2\Theta}{d\theta^2}+3\cot\theta\frac{d\Theta}{d\theta}\right)=-S,$$

$$(16)\quad 2\sin^2\chi\left(-\frac{2\kappa Mc^2}{3\pi^2 a}-\frac{1}{3}\lambda a^2+3\right)$$
$$+\frac{\sin^2\chi}{X}\left(\frac{d^2X}{d\chi^2}+4\cot\chi\frac{dX}{d\chi}-4X\right)=S.$$

Equation (15) has a regular solution for $\theta=0$ and π only when S= (n+2) (n−1), where n is an integer, and which is the first derivative of the Legendre polynomials $Pn(\cos\theta)$. As the particular solution (13) does not involve the variable θ, function $\Theta(\theta)$ must be constant in order that the solution $\omega(\chi,\theta,t)$ can connect smoothly at $\chi=\chi_0$ and $\chi=\chi_1$. Therefore n=1 and S=0.

Equation (10) with $T(t)\to 0$ means that the variable χ and t must separate in Eq. (16) (S=0), therefore next conditions must be satisfied for consistency:

$$(17)\quad 2\left(\frac{2\kappa Mc^2}{3\pi^2 a}+\frac{1}{3}\lambda a^2-3\right)=D=\text{const.},$$

$$(18)\quad \frac{1}{X}\left(\frac{d^2X}{d\chi^2}+4\cot\chi\frac{dX}{d\chi}-4X\right)=D.$$

Due to the condition (17), the particular solution (13) is rewritten to

$$(19)\quad \omega_p=\frac{\kappa Mc^2}{\pi^2(D+4)a(t)}\omega_s$$

If we use a variable z defined by $2z=\cos\chi+1$, equation (18) reduces to the hypergeometric differential equation:

$$(20)\quad z(1-z)\frac{d^2X}{dz^2}+\left(\frac{5}{2}-5z\right)\frac{dX}{dz}-(D+4)X=0.$$

The independent solution in the vicinity of $z=0$ are the hypergeometric functions

$$(21)\quad F(\alpha,\beta,\gamma;z),\quad z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z),\quad |z|<1,$$

and in the vicinity of $z=1$

(22) $F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z),$

$$(1-z)^{\gamma-\alpha-\beta}F(\alpha-\beta,\gamma-\alpha,\gamma-\alpha-\beta+1;1-z),\quad |1-z|<1,$$

where

(23) $\alpha+\beta=4,\quad \alpha\beta=D+4,\quad \gamma=\frac{5}{2}$

The function ω must be regular at $\chi=0$ and $\chi=\pi$. Thus, the complete solution $\omega(\chi,\theta,t)$ is

$$(24)\quad \begin{cases} \omega_a(\chi,t)=\dfrac{A}{a^3(t)}X_a(\chi), & (0\leqq\chi\leqq\chi_0), \\ \omega_b(\chi,t)=\dfrac{1}{a^3(t)}[B_1X_{b1}(\chi)+B_2X_{b2}(\chi)]+\dfrac{\kappa Mc^2}{\pi^2(D+4)a}\omega_s, & (\chi_0\leqq\chi\leqq\chi_1), \\ \omega_c(\chi,t)=\dfrac{C}{a^3(t)}X_c(\chi), & (\chi_1\leqq\chi\leqq\pi), \end{cases}$$

where

$$(25)\quad \begin{aligned} &X_a(\chi)=F(\alpha,\beta,\alpha+\beta-\gamma+1;(1-\cos\chi)/2), \\ &X_{b1}(\chi)=X_c(\chi)=F(\alpha,\beta,\gamma;(1+\cos\chi)/2), \\ &X_{b2}(\chi)=\left(\frac{1+\cos\chi}{2}\right)^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;(1+\cos\chi)/2), \end{aligned}$$

and A, B_1, B_2, and C are arbitrary constants.

These constants are determined by means of the conditions that ω_a, ω_b, and ω_c must connect smoothly at $\chi=\chi_0$ and $\chi=\chi_1$, that is,

$$(26)\quad \begin{cases} \omega_a(\chi_0,t)=\omega_b(\chi_0,t), \\ \omega_b(\chi_1,t)=\omega_c(\chi_1,t), \\ \partial_\chi\omega_a(\chi_0,t)=\partial_\chi\omega_b(\chi_0,t), \\ \partial_\chi\omega_b(\chi_1,t)=\partial_\chi\omega_c(\chi_1,t). \end{cases}$$

As the rotating shell obeys the ordinary conservation law of angular momentum, the angular volocity ω_s varies in proportion to the inverse square of a(t). Therefore the particular solution ω_p varies in proportion to the inverse cube of a(t), and there arise no contradictions in Eqs. (26) for all t.

Now we are interested in the metric tensor in the vicinity of the origin of the universe, so it is enough to determine only the value of A in Eqs. (24). By solving simultaneously Eqs. (26) we have

(27) $A = Q(\chi_0,\chi_1)/P(\chi_0,\chi_1)$ (see Appendix B1),

and hence the solution inside the shell is

$$(28)\quad \omega_a(\chi_0,\chi_1;\chi,t) = \frac{Q(\chi_0,\chi_1)}{P(\chi_0,\chi_1)}\cdot\frac{1}{a^3(t)}\cdot \times F(\alpha,\beta,\alpha+\beta-\gamma+1;\,(1-\cos\chi)/2).$$

At the origin

$$(29)\quad \omega_0(\chi_0,\chi_1;t) = \lim_{\chi\to 0}\omega_a = \frac{Q(\chi_0,\chi_1)}{P(\chi_0,\chi_1)}\cdot\frac{1}{a^3(t)}.$$

This function represents the the angular velocity of the inertial frame at the origin of the Friedmann universe, induced by the spherical rotating shell restricted by two hypersurfaces $\chi=\chi_0$ and $\chi=\chi_1$ with the angular velocity ω_s.

The induced angular velocity in case that the rotating shell covers the whole universe coverges to

$$(30)\quad \omega_0(t) = \lim_{\chi_0\to 0}\lim_{\chi_1\to\pi}\omega_0(\chi_0,\chi_1;t) = \omega_p \qquad \text{(see Appendix B2)}.$$

Therefore the dragging coeffcient of the inertial frame in the Friedmann universe is given as

$$(31)\quad \frac{\omega_0}{\omega_s}=\frac{8}{\pi(D+4)}\cdot\frac{GM}{c^2a(t)}.$$

Ⅲ. Discussions and Concluding Remarks

Equation (31) means that the dragging coefficient of the inertial frame decreases as the universe expands. This result coinsides with the intuitive prediction from Thirring's result, and reconfirms importance of the factor GM/Rc^2 in the inertial frame dragging. The inertial frame is not completely connected with matter of the universe, therefore the Coriolis force does not appear completely by the rotation relative to the whole universe. The Friedmann universe is not Machian in this sence. We cannot help thinking that the inertial force does not have the material origin in general relativity, and is introduced *a priori* to the theory.

There exists a difficulty in the present discussion. We cannot understand the physical meaning of the condition (17), bseides we cannot determine a value of D in the framework of the theory. We cannot give a fixed value of the dragging coefficient in the Friedmann universe.

If we put the relation $\omega_0/\omega_s=1$ from the Machian point of view, we obtain two relations $G/a=\text{const.}$ and $\lambda a^2=\text{const.}$, which lead us to the static Einstein universe. If we assume the relation $\omega_0/\omega_s=1$, and moreover request the expanding universe, we cannot help extending the theory. We need the variable gravitational "constant" and the variable cosmological "constant". The time dependence of those "constants" is clear, that is, $G(t)\propto a(t)$ and $\lambda(t)\propto a^{-2}(t)$.

The Machian cosmological model satisfying those conditions exists in the Brans-Dicke theory of gravitation. The gravitational field, in this theory, is described by the metric tensor and the scalar field in the Riemannian manifold which represents the reciprocal of the gravitational "constant". The universe expands forever linearly $a(t) \propto t$, and the scalar field satisfies the relation $a(t)\phi(t) = \text{const.}$ for all cosmological times. The cosmological "constant" is regarded as $\lambda(t) \equiv -(\eta/2)(\dot{\phi}/\phi)^2 \propto a^{-2}(t)$ in this model[9]. The correspondent term to Eq. (17) becomes constant automatically and its value is given as the result of the theory. Thus, the relation $\omega_0/\omega_s = 1$ is always statisfied. Now we can understand how difficulties in the Friedmann universe have solved in this closed cosmological model of the Brans-Dicke theory.

In the present discussion, the (3, 0) component of the field equations, especially the particular solution of this inhomogeneous equation, determines the inertial property of the universe ($\omega_0 \rightarrow \omega_p$). The particular solution is $\omega_p = \omega_s$ in that model of the Brans-Dicke theory, and $\omega_p \propto \omega_s \cdot a^{-1}(t)$ in the Friedmann universe. This is why the dragging coefficient of the inertial frame ω_0/ω_s decreases as $a^{-1}(t)$ in the Friedmann universe.

For the present, the expanding Friedmann universe is the most standard cosmological model in general relativity. Therefore, even if the Friedmann universe becomes non-Machian in this investigation, much more study should be tried to confirm this result.

Since Newton, all (local) theories of physics have been described in reference to the absolute space or the global inertial frame, which is never influenced by the environment (the universe). It is sure that Einstein reduced the global inertial frame to the local in general relativity, in which the gravitational field, that is, the

Riemannian space-time is subject to the distribution of matter. However, is the local inertial frame really determined completely by the distribution of matter in the universe ? The inertial force appeared in the coordinate transformation of references is nothing but the fictitious. Is it to matter of the universe or to the absolute space that the inertial force appears in the acceleration ? It seems that a ghost of the absolute space remains even in general relativity.

Again we ask what the inertial frame is. Our aim is to give inertia the material origin.

Appendix A1.

nonvanishing covariant components of the metric tensor

$g_{11}=a^2(t)$

$g_{22}=a^2(t)\sin^2\chi$

$g_{33}=a^2(t)\sin^2\chi\sin^2\theta$

$g_{00}=-1$

$g_{30}=g_{03}=-(1/c)a^2(t)\sin^2\chi\sin^2\theta\,\omega(\chi,\theta,t)$

nonvanishing contravariant components of the metric tensor

$g^{11}=1/a^2$

$g^{22}=1/a^2\sin^2\chi$

$g^{33}=1/a^2\sin^2\chi\sin^2\theta$

$g^{00}=-1$

$g^{30}=g^{03}=-\omega/c$

Appendix A2.

nonvanishing Christoffel symbols

$\Gamma^1_{10}=\Gamma^1_{01}=\dot{a}/a$

$\Gamma^1_{22}=-\sin\chi\cos\chi$

$\Gamma^1_{33}=-\sin\chi\cos\chi\sin^2\theta$

$\Gamma^1_{30}=\Gamma^1_{03}=(\omega/c)\sin\chi\cos\chi\sin^2\theta+(1/2c)\sin^2\chi\sin^2\theta\,\partial_\chi\omega$

$\Gamma^2_{12}=\Gamma^2_{21}=\cot\chi$

$\Gamma^2_{20}=\Gamma^2_{02}=\dot{a}/a$

$\Gamma^2_{33}=-\sin\theta\cos\theta$

$\Gamma^2_{30}=\Gamma^2_{03}=(\omega/c)\sin\theta\cos\theta+(1/2c)\sin^2\theta\,\partial_\theta\omega$

$\Gamma^3_{11}=(\omega/c)a\dot{a}$

$\Gamma^3_{13}=\Gamma^3_{31}=\cot\chi$

$\Gamma^3_{10}=\Gamma^3_{01}=-(\omega/c)\cot\chi-(1/2c)\partial_\chi\omega$

$\Gamma^3_{22}=(\omega/c)a\dot{a}\sin^2\chi$

$\Gamma^3_{23}=\Gamma^3_{32}=\cot\theta$

$\Gamma^3_{20}=\Gamma^3_{02}=-(\omega/c)\cot\theta-(1/2c)\partial_\theta\omega$

$\Gamma^3_{33}=(\omega/c)a\dot{a}\sin^2\chi\sin^2\theta$

$\Gamma^3_{30}=\Gamma^3_{03}=\dot{a}/a$

$\Gamma^3_{00}=-(2\omega/c)\cdot(\dot{a}/a)-\dot{\omega}/c$

$\Gamma^0_{11}=a\dot{a}$

$\Gamma^0_{13}=\Gamma^0_{31}=(1/2c)a^2\sin^2\chi\sin^2\theta\,\partial_\chi\omega$

$\Gamma^0_{22}=a\dot{a}\sin^2\chi$

$\Gamma^0_{23}=\Gamma^0_{32}=(1/2c)a^2\sin^2\chi\sin^2\theta\,\partial_\theta\omega$

$\Gamma^0_{33}=a\dot{a}\sin^2\chi\sin^2\theta$

$\Gamma^0_{30}=\Gamma^0_{03}=-(\omega/c)a\dot{a}\sin^2\chi\sin^2\theta$

Appendix A3.

nonvanishing components of the Ricci tensor

$R_{11}=-2-2\dot{a}^2-a\ddot{a}$

$R_{22}=(-2-2\dot{a}^2-a\ddot{a})\sin^2\chi$

$R_{33}=(-2-2\dot{a}^2-a\ddot{a})\sin^2\chi\sin^2\theta$

$R_{00}=3\ddot{a}/a$

$R_{13}=R_{31}=-(1/2c)a^2\sin^2\chi\sin^2\theta\,[\partial_\chi\dot{\omega}+3(\dot{a}/a)\partial_\chi\omega]$

$R_{23}=R_{32}=-(1/2c)a^2\sin^2\chi\sin^2\theta\,[\partial_\theta\dot{\omega}+3(\dot{a}/a)\partial_\theta\omega]$

$R_{30}=R_{03}=(\omega/c)\sin^2\chi\sin^2\theta(a\ddot{a}+3\dot{a}^2)$

$\quad-(1/2c)\sin^2\chi\sin^2\theta(\partial^2_{\chi\chi}\omega+4\cot\chi\,\partial_\chi\omega-4\omega)$

$\quad-(1/2c)\sin^2\theta(\partial^2_{\theta\theta}\omega+3\cot\theta\,\partial_\theta\omega)$

the scalar curvature

$R=-(6/a^2)(1+\dot{a}^2+a\ddot{a})$

Appendix A4.

nonvanishing components of the Einstein tensor

$G_{11}=1-\lambda a^2+\dot{a}^2+2a\ddot{a}$

$G_{22}=G_{11}\sin^2\chi$

$G_{33}=G_{11}\sin^2\chi\sin^2\theta$

$G_{00}=-(3/a^2)(1+\dot{a}^2)+\lambda$

$G_{13}=G_{31}=-(1/2c)a^2\sin^2\chi\sin^2\theta\,[\partial_\chi\dot{\omega}+3(\dot{a}/a)\partial_\chi\omega]$

$G_{23}=G_{32}=-(1/2c)a^2\sin^2\chi\sin^2\theta\,[\partial_\theta\dot{\omega}+3(\dot{a}/a)\partial_\theta\omega]$

$G_{30}=G_{03}=-(\omega/c)\sin^2\chi\sin^2\theta(3-\lambda a^2+2a\ddot{a})$

$\quad-(1/2c)\sin^2\chi\sin^2\theta(\partial^2_{\chi\chi}\omega+4\cot\chi\,\partial_\chi\omega-4\omega)$

$\quad-(1/2c)\sin^2\theta(\partial^2_{\chi\chi}\omega+3\cot\theta\,\partial_\theta\omega)$

Appendix B1.

$$P(\chi_0,\chi_1)=\begin{vmatrix} X_a(\chi_0) & X_{b_1}(\chi_0) & X_{b_2}(\chi_0) & 0 \\ 0 & X_{b_1}(\chi_1) & X_{b_2}(\chi_1) & X_c(\chi_1) \\ X'_a(\chi_0) & X'_{b_1}(\chi_0) & X'_{b_2}(\chi_0) & 0 \\ 0 & X'_{b_1}(\chi_1) & X'_{b_2}(\chi_1) & X'_c(\chi_1) \end{vmatrix},$$

$$Q(\chi_0,\chi_1)=\begin{vmatrix} \Omega & X_{b_1}(\chi_0) & X_{b_2}(\chi_0) & 0 \\ \Omega & X_{b_1}(\chi_1) & X_{b_2}(\chi_1) & X_c(\chi_1) \\ 0 & X'_{b_1}(\chi_0) & X'_{b_2}(\chi_0) & 0 \\ 0 & X'_{b_1}(\chi_1) & X'_{b_2}(\chi_1) & X'_c(\chi_1) \end{vmatrix},$$

where

$$\Omega=\frac{\kappa Mc^2}{\pi^2(D+4)}\cdot a^2\omega_s.$$

Appendix B2.

asymptotic behavior of the hypergeometric function

$$X_a(0)=1$$

$$X'_a(0)=0,\ X'_a|(\chi)=\frac{\alpha\beta}{2(\alpha+\beta-\gamma+1)}\sin\chi \qquad (\chi\sim 0)$$

$$X_{b_2}(0)=+\infty,\ X_{b_2}(\chi)=\left(\frac{1-\cos\chi}{2}\right)^{\gamma-\alpha-\beta}\cdot$$

$$\times\frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} \qquad (\chi\sim 0)$$

$$X_{b_2}(\pi)=+\infty,\ X_{b_2}(\chi)=\left(\frac{1+\cos\chi}{2}\right)^{1-\gamma} \qquad (\chi\sim\pi)$$

$$X'_{b_2}(0)=-\infty,$$

$$X'_{b_2}(\chi)=\frac{\gamma-\alpha-\beta}{2}\sin\chi\left(\frac{1-\cos\chi}{2}\right)^{\gamma-\alpha-\beta-1}$$

$$\times\frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} \qquad (\chi\sim 0)$$

$$X'_{b_2}(\pi) = +\infty,\ X'_{b_2}(\chi) = \frac{\gamma-1}{2}\sin\chi\left(\frac{1+\cos\chi}{2}\right)^{-\gamma} \qquad (\chi\sim\pi)$$

$$X_c(0) = +\infty,\ X_c(\chi) = \left(\frac{1-\cos\chi}{2}\right)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \qquad (\chi\sim 0)$$

$$X_c(\pi) = 1$$

$$X'_c(0) = -\infty,$$

$$X'_c(\chi) = \frac{\gamma-\alpha-\beta}{2}\sin\chi\left(\frac{1-\cos\chi}{2}\right)^{\gamma-\alpha-\beta-1}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \qquad (\chi\sim 0)$$

$$X'_c(\pi) = 0,\ X'_c(\chi) = \frac{\alpha\beta}{2\gamma}\sin\chi \qquad (\chi\sim\pi)$$

$\Gamma(z)$ is the gamma function.

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