New Foundations of General Relativistic Theory of Gravitation

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Dedicated to A. Einstein

Abstract

The material origin of inertia is discussed on new foundations of general relativistic theory of gravitation, the general principle of local relativity and the principle of inertia. The field equtions are Brans-Dicke-type, and the law of force is presumed. The proper inertial frame is rest to the distribution of matter in the universe. The equation of motion in the gravitational field can be obtained from motion of a free-particle in the smoothed-out universe, and the principle of equivalence is derived as a result of the theory. The inertial force is induced by the acceleration to matter in the universe, and inertial mass of a particle is given by its gravitational mass and the cosmological scalar field. No particles have inertia in otherwise empty space. Einstein's conjecture on inertia is dis-

cussed briefly. The distinction between global and local observers is essential to make the origin of inertia clear.

I. Introduction

Problems of space-time and gravitation have been puzzling and fascinating to mankind for a long time.

In the Newton theory, the gravitational field is described by a gravitational potential ϕ,

$$\boldsymbol{F} = -m \text{ grad } \phi, \tag{1}$$

which is determined by the distribution of mass density through the Poisson equation. Motion of a test-particle in the gravitational field is determined by the equation of motion

$$m \frac{d^2 \boldsymbol{r}}{dt^2} = \boldsymbol{F}, \tag{2}$$

which is valid in reference to absolute space or to inertial frames which do not accelerate to it. The inertial force appears in reference frames accelerating to absolute space. Absolute space is not influenced by the environment in the universe, and is the absolute reference frame of motion. Newton cited the bucket gedanken experiment to show the absoluteness of motion. He thought that a change of the surface of water in the bucket is not due to the rotation of water relative to the bucket but to the rotation to absolute space.

According to Mach, only relative motion to other bodies is detectable. He noticed the existence of matter in the universe behind the bucket. What happens in the rotating bucket if its wall becomes much thicker and more massive? He insisted that the appearence of the Coriolis and the centrifugal forces is due to the rotation relative to matter in the universe. The inertial frame is dominated by the distribution of matter in the universe.

The Einstein theory is constructed by means of two fundamental principles, the general principle of relativity, and the strong principle of equivalence. The former means all reference frames are undistinguished, and requests all laws of physics are generally covariant. The latter means an inertial frame can always be constructed in the small region for a given gravitational field, and the equation of motion of an infinitesimally small test-particle is that of geodesic. In other words the gravitational and the inertial forces are locally equivalent.

The gravitational field is represented by the Riemannian manifold. The metric tensor is calculated from the energy-momentum tensor through the Einstein equations. The local inertial frame is determined by the distribution of matter in a meaning. However, the origin of inertia is not clear in the Einstein theory. The inertial force is introduced *a priori* in the elevator gedanken experiment. It is obscure what the inertial force is induced by the acceleration to. A test-particle also has inertia in Minkowski space in which matter does not exist. The theory includes anti-Machian metric like the Gödel universe.

In the Brans-Dicke theory[1)] the gravitational field is described by the Riemannian manifold and the scalar field ϕ on it. The weak principle of equivalence is assumed, and the gravitational "constant" is allowed to vary in space and time. The equation of motion of a test-particle is geodesic. On the origin of inertia, the Brans-Dicke theory has similar defects in the Einstein theory. The scalar field is generally expanded as following

$$\phi=\phi_0+O(\rho/\eta)+\cdots. \tag{3}$$

The constant ϕ_0 is correspondent to the gravitational constant G in the Einstein theory in the limit $|\eta|\to\infty$. This means that even empty space gives inertia.

D. W. Sciama proposed a simple theory[2] on the origin of inertia. His ideas consist of two main stage, that is, a theory-model of gravitation and a cosmological model. In his theory the gravitational field is described by a scalar ϕ and a vector potential $\boldsymbol{A}$, which obey to Maxwell-type field equations, and produce a following force

$$\boldsymbol{E}=-\mathrm{grad}\,\phi-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}. \tag{4}$$

As the equation of motion, he introduced the postulate that *in the rest-frame of any body the total gravitational field at the body arising from all the other matter in the universe is zero.*

By applying this theory to a problem, motion of a falling test-particle in the gravitational field created by the Sun and the background universe, he obtained the following equation of motion:

$$-\frac{\phi_U+\phi_m}{c^2}\cdot\frac{dv}{dt}=-\frac{m}{r^2}, \tag{5}$$

where ϕ_U, ϕ_m are scalar potentials created by the universe and the Sun respectively, m is mass of the Sun, and $+v$ is a velocity of the particle to the background universe. This equation determines the local inertial frame at the point of the particle through the re-definition of the inertial frame, that is, the rest-frame of the particle accelerating relatively to the universe. The right-hand side of equation (5) is the gravitational force by the Sun, and the left-hand side represents the inertial force created by the acceleration relative to the universe. They are equivalent to one another, and this is nothing but the principle of equivalence.

Sciama took as his background cosmological model to support the scalar field a homogeneous and isotropic distribution of matter of density ρ expanding uniformly according to the Hubble law. Assuming the matter receding with velocity greater than that of light makes no contribution to the potential, he got as the scalar potential

$$\phi_U = -2\pi\rho c^2\tau^2, \tag{6}$$

where $c\tau$ is Hubble radius. As the universe expands, the density ρ descreases in proportion to the inverse cube of τ. Thus the scalar potential ϕ_U decreases in inverse-proportion to τ. Since the gravitational constant satisfies $G\phi_U = -c^2$, the equation (6) implies that $G\rho\tau^2 \sim 1$ or

$$GM/Rc^2 \sim 1, \tag{7}$$

where M is mass of the total universe, and R is Hubble radius.

Sciama's theory was a fascinating attempt for Mach's ideas, but it is unsatisfactory concerning two points: The gravitational field must be tensor-type, and his cosmological model is lacking for relativistic treatment.

In this paper a general relativistic theory of gravitation is developed on new foundations to clarify the material origin of inertia. This theory is bassed on two fundamental principles, that is, the general principle of local relativity, and the principle of inertia. The field equations which describe the gravitational interaction are formally same as those of the Brans-Dicke theory. The essential difference of viewpoint to ordinary theories of gravitation exists in distinguishing local and global observers. We premise the distribution of matter in the universe and construct the global theory of gravitation.

II. Physical Foundations of the Theory

global and local observers

We consider two kinds of observers. One is *an observer in a laboratory with windows*, who can look out over the universe. Let us call him a global observer. The other is *an observer in a laboratory without windows*. There is nothing for him but to construct his theory by means of only local phenomena in his elevator. We call him a local observer. It is, of course, a global observer that can construct essential theories of physics.

the general principle of local relativity

All reference frames are indistinguishable for local observers.

For global observers, the rest-frame of the universe is a special reference frame. They may use it as the absolute reference frame of motion. We, however, may also insist that motion is relative to the last. In saying that a reference frame is accelerating to the rest-frame of the universe, we must add "relatively" to be exact. We would need again another absolute reference frame to determine which frame accelerates absolutely. We can not suppose such a reference frame. Physical meaning of relativity will be discussed later.

At any rate, mathematically, *all laws of physics must be expressed in the covariant form to arbitrary coordinate transformations.*

the proper local inertial frame

It is defined as *a reference frame to which infinitesimally small test-particles receiving no external forces including local gravitational forces in the smoothed-out distribution of matter in the universe do not accelerate.*

We suppose that these free-particles are influenced from matter in the universe. Generally this frame is restricted to small region compared with the whole universe. It is clear for global observers whether local gravitational and other external forces act on test-particles or not.

the local inertial frame

It is re-defined as *a reference frame to which infinitesimally small test-particles receiving no external forces except local gravitational forces in a given external gravitational field do not accelerate.*

It will be showed, in the result of the theory, that a local inertial frame can always be constructed at any point for an arbitrary gravitational field.

the principle of inertia

The proper local inertial frame does not accelerate globally to the smoothed-out distribution of matter in the universe.

We know empirically that the inertial frame does not rotate to distant stars. We exalt this fact to a principle, and construct a theory of gravitation on this base. This principle also has a role as a selection rule, and excludes anti-Machian metric like the Gödel universe. If there exists no matter in the universe, this principle loses its physical meaning, and the theory is ruined automatically.

From the definition of the proper inertial frame and the principle of inertia, a free-particle in the smoothed-out universe does not accelerate to the rest-frame of the universe at any point. Therefore motion of this free-particle is described by the following equation:

$$\frac{d^2x^\mu}{d\tau^2}=0, \tag{8}$$

where x^μ is a coordinate system fixed to matter in the universe, and τ is proper-time in the rest-frame of the universe. The equation of motion of a test-particle in the given gravitational field is derived from this equation.

the field equations of gravitation

The gravitational field, in this theory, is described by the metric tensor of the Riemannian manifold and the scalar field on it as well as in the Brans-Dicke theory. The field equations are derived from the variational principle

$$\delta\int[\phi R+(16\pi/c^4)L-\eta(\phi_{,\mu}\phi'^{\mu}/\phi)](-g)^{1/2}d^4x=0, \tag{9}$$

where L is the Largrangian density of matter and η is a coupling parameter of the scalar field. We obtain

$$\begin{cases} R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{8\pi}{c^4\phi}T_{\mu\nu}-\frac{\eta}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi'^{\lambda}\right) \\ \qquad\qquad -\frac{1}{\phi}(\phi_{,\mu;\nu}-g_{\mu\nu}\Box\phi), & \text{(10a)} \\ \Box\phi=-\frac{8\pi}{(3+2\eta)c^4}T, & \text{(10b)} \end{cases}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter.

The physical meaning of the scalar field ϕ is cleared up in correspondence to other theories. In correspondence to the Newton theory, it may be interpreted as the gravitational "constant" or inertial mass. In the Brans-Dicke theory the scalar field represents the reciprocal of the gravitational "constant", and motion of a test-particle in a given gravitational field becomes geodesic. If inetial mass varies in space and time, the equation of motion is no more geodesic.

In correspondence to the Sciama theory, we may also interpret that the scalar field represents inertial mass. You will find this interpretation makes the material origin of inertia clearer. It is remarkable that even if inertial mass varies in space and time, motion of a test-particle becomes geodesic in the present theory.

the law of force

The force generated by the given external gravitational field to an infinitesimally small test-particle is formulated as

$$F^{\mu}=-\phi\Gamma^{\mu}_{\nu\lambda}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau}, \tag{11}$$

where a parameter τ must be the same as that of the four-acceleration $d^2x^{\mu}/d\tau^2$. Let us always use proper-time in the rest-frame of the universe as this parameter, and call it *cosmic time*. As the scalar field ϕ is almost determined cosmologically by the distribution of matter in the universe, we may put ϕ_U in Eq. (11).

The gravitational field produces only this unique force. The concept of the inertial and the gravitational forces is introduced later, but their root is same.

The validity of the above formulation, as well as the field equations, should be testified by experiments or observations. In the weak-field approximation the formulation (11) is reduced to Eq.

(1) in the Newton theory (except a numerical factor).

III. The Background Universe

All phenomena arise in our real universe. All physical phenomena would be influenced by the distribution of matter in the universe to be exact. Unfortunately, however, we do not know the whole universe yet.

We presume a cosmological model[3)4)5)]. This closed cosmological model satisfies the field equations for the homogeneous and isotropic distribution of matter, and has some interesting properties[4)5)6)7)8)] in the Machian point of view.

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{12}$$

$$\begin{cases} a(t) = [-2/(2+\eta)]^{1/2}t, \qquad (\eta < -2) & \text{(13a)} \\ 2\pi^2 a^3(t)\rho(t) = M, & \text{(13b)} \\ \phi_U(t) = -[8\pi/(3+2\eta)c^2]\rho(t)\cdot t^2. & \text{(13c)} \end{cases}$$

In this model the following relation is always satisfied:

$$G(t)M/c^2a(t) = \pi, \tag{14}$$

where $G \equiv (4+2\eta)/(3+2\eta)\phi$. You will find this general relativistic cosmological model has good correspondence to Sciama's model.

The constant scalar field ϕ_0, which is correspondent to the *a priori* gravitational constant G, does not exist in this particular model. In the limit $\rho \to 0$, the cosmological scalar field also tends to zero. For this result the present theory does not give inertia to a test-particle in otherwise empty space.

It is obvious that the principle of inertia is automatically satisfied in this cosmological model. All galaxies in the universe do not accelerate to each other and the inertial frame at the origin of the universe does not accelerate to them. In the Friedmann universe, the second derivative of the expansion parameter $a(t)$ does not vanish, and distant galaxies are accelerating to the Earth in radial direction.

Ⅳ. Induction of the Inertial Field

(i) motion of a free-test-particle in a reference frame accelerating to the rest-frame of the universe.

We consider the accelerating part (elevator) of this reference frame is restricted to small region and rests instantly to the rest-frame of the universe. Let us put coordinates x'^{μ} to this reference frame covering the whole universe. For global observers, two coordinate system are lelated as

$$x^{\mu}=x'^{\mu}+\xi^{\mu}, \qquad (15)$$

where $\xi^0\equiv 0$, $d^2\xi^i/d\tau^2=$const. in the elevator, and ξ^i decreases smoothly outside and vanishes asymptotically. Notice we always use cosmic time as a parameter.

In the present theory only the equation of motion of a free-particle in the rest-frame of the universe is given. We get directly

$$\frac{d^2x^{\mu}}{d\tau^2}=\frac{d^2x'^{\mu}}{d\tau^2}+\frac{d^2\xi^{\mu}}{d\tau^2}=0. \qquad (16)$$

This means the test-particle moves with the acceleration $-d^2\xi^{\mu}/d\tau^2$ in the new reference frame.

According to the general principle of local relativity, the equation of motion is also expressed in general covariant way. Therefore Eq. (8) must be generally described by means of the covariant derivative

$$\frac{D^2x^\mu}{D\tau^2}=0. \tag{17}$$

In the new coordinate system the equation of motion of the free-test-particle is written down explicitly

$$\frac{d^2x'^\mu}{d\tau^2}+\Gamma'^\mu_{\nu\lambda}\frac{dx'^\nu}{d\tau}\ \frac{dx'^\lambda}{d\tau}=0. \tag{18}$$

This expression is due to a mere coordinate transformation, and has no special physical meanings. In fact, motion of the free-particle is nothing but uniform rectilinear motion to matter of the universe.

We obtain

$$\Gamma'^\mu_{\nu\lambda}\frac{dx'^\nu}{d\tau}\ \frac{dx'^\lambda}{d\tau}=\frac{d^2\xi^\mu}{d\tau^2} \tag{19}$$

immediately from Eqs. (16) and (18), or from the transformation property of the affine connection

$$\Gamma^\mu_{\nu\lambda}{}'(x')=\frac{\partial x'^\mu}{\partial x^\rho}\ \frac{\partial x^\alpha}{\partial x'^\nu}\ \frac{\partial x^\beta}{\partial x'^\lambda}\Gamma^\rho_{\alpha\beta}(x)-\frac{\partial^2x'^\mu}{\partial x^\alpha\partial x^\beta}\ \frac{\partial x^\alpha}{\partial x'^\nu}\ \frac{\partial x^\beta}{\partial x'^\lambda}. \tag{20}$$

It is a problem of the field equations how the gavitational field is generated by the given distribution of matter in the present coordinate system. We can find the metric tensor $g_{\mu\nu}$ by writing down the energy-momentum tensor $T_{\mu\nu}$ and solving the field equations. It is given by the law of force how much force is produced by the gravitational field.

On the other hand, as the field equations are generally covariant, the gravitational field solved directly in the new coordinate system is same as that by the coordinate transformation from the rest-frame of the universe. However, to be exact, the solution is not determined only by the field equations but also by boundary conditions. In the present case, as the coordinate transformation is restricted to small region, so the boundary conditions remain same, and the gravitational field obtained by the coordinate transformation surely satisfies them. Therefore the force generated by the gravitational field in the new reference frame is given as

$$F'^{\mu}=-\phi_U\Gamma'^{\mu}_{\nu\lambda}\frac{dx'^{\nu}}{d\tau}\;\frac{dx'^{\lambda}}{d\tau}=-\phi_U\frac{d^2\xi^{\mu}}{d\tau^2}. \tag{21}$$

In this way we find a kind of the gravitational field is induced in the reference frame accelerating to the rest-frame of the universe. In the concept of the Newton theory, the force F'^{μ} is correspondent to the inertial force. Therefore the induced gravitational field can be regard as the inertial field. Global observers in this reference frame understand that by means of the inertial force F'^{μ} the free-test-particle moves with the acceleration $-d^2\xi^{\mu}/d\tau^2$ to their frame. However, the free-particle practically does not accelerate to matter of the universe, and only the reference frame merely accelerates to it. Let us call them *the apparent inertial force* and *field* repectively. Though we call them "*apparent*", they arise as the solution of the field equations, and so are the actual field and force in this reference frame.

In the viewpoint of relativity of motion, we may regard that matter of the universe accelerates to the observer in the elevator as a whole. In this case we interpret that the inertial field is induced in this reference frame by means of physical mechanism of the field equations, "*inertial frame dragging*". We will briefly discuss this effect later. Anyhow, in either case, it is the distribution of matter in the universe that the inertial field is induced by the relative acceleration to.

(ii) motion of a test-particle in a given local gravitational field

We consider a case that there exist locally restricted massive bodies in the smoothed-out background universe, and a test-particle practically accelerates to matter of the universe by their attraction.

In the rest-frame of the test-particle, from its meaning, the equation of motion is described as

$$\frac{d^2x'^{\mu}}{d\tau^2}=0. \tag{22}$$

Notice that this equation does not mean the rest-frame of a given test-particle directly becomes the local inertial frame at its point.

Let us see this motion from the rest-frame of the universe. We obtain by means of a mere coordinate transformation

$$\frac{d^2x^{\mu}}{d\tau^2}+\Gamma^{\mu}_{\nu\lambda}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau}=0. \tag{23}$$

The gravitational field produced by local bodies in the rest-frame of the universe is obtained by solving the field equations with boud-dary conditions, and coinsides with the solution derived by the co-ordinate transformation due to the general principle of local relati-vity. Thus the local gravitational force acting on the test-particle from this gravitational field is given as

$$F^{\mu}=-\phi_U\Gamma^{\mu}_{\nu\lambda}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau}. \tag{24}$$

For global observers in the rest-frame of the universe, the test-particle moves with the acceleration $d^2x^{\mu}/d\tau^2$ to the distribution of matter in the universe by means of this local gravitational attrac-tion.

Again we return to the rest-frame of the particle. As this re-ference frame accelerates to the distribution of matter in the uni-verse, the inertial force is induced:

$$F'^{\mu} = -\phi_U \frac{d^2 x^{\mu}}{d\tau^2}. \tag{25}$$

Therefore global observers in the rest-frame of the particle understand the equation (22) is derived as a result that the induced inertial force balanced the local gravitational force acting on the test-particle, for they can recognize both forces independently

The above discussion does not include particular properties of the test-particle. Global observers in the rest-frame of the universe find that all test-particles are received the same acceleration from the given local gravitational force, and those in the rest-frame of the particle find that all local gravitational forces acting on other test-particles also vanish respectively at the same time in this frame. This is nothing but the principle of equivalence. It has been derived as a result of the present theory. The rest-frame of the particle, that is, the freely-falling frame is surely the local inertial frame at its point.

Originally the gravitational force and the inertial force are essentially same, and have the same material origin, that is, matter of the universe. For local observers both forces are indistinguishable except an ordinary remark as well as in the Einstein theory. In the above discussion, unfortunate local observers in the rest-frame of the universe may think that the local gravitational force does not act on the test-particle, but they are in an accelerating reference frame. We wonder they think to what the inertial force appears by the acceleration.

In order to discuss the material origin of inertia, it is extremely important to distinguish global and local observers. In past ordinary theories in which only local observers are taken into account essentially, the origin of inertia remains vague. Existence of this ambiguity becomes much clearer when we discuss empty space.

Let us call the inertial force induced to the particle in the rest-frame of the particle when it actually accelerates to matter of the universe the "*proper*". We understand this proper inertial force is generated by the interaction between the accelerating particle and matter of the universe. The starting-point Eq. (8) means that the proper inertial force does not appear to a free-particle in the smoothed-out universe. When an external force acts on the particle, the proper inertial force balances it in the rest-frame of the particle. Thus the particle maintains its state (acceleration) to the distribution of matter in the universe.

After all the equation of motion for infinitesimally small test-particles receiving no external forces (except gravitation) is described as

$$\frac{D^2x^\mu}{D\tau^2}=0, \tag{26}$$

in arbitrary reference frames, in the given gravitational field. Taking the physical meaning into account we may as well adopt

$$\phi_U\frac{D^2x^\mu}{D\tau^2}=0 \tag{27}$$

as the equation of motion. When matter exists in the universe, the cosmological background scalar field does not vanish ($\phi_U\neq 0$), and the equation (27) reduces to that of geodesic.

(iii) motion of a freely-falling test-particle to the Sun in the background universe

Within the framework of the theory of gravitation, the concept of inertial mass is not always necessary. However, we are interested in the correspondent physical meaning in the Newton or the Sciama theories.

Let us investigate motion of a test-particle receiving the local gravitational force generated by the Sun in the present cosmological model as the background universe. The region where we consider local motion is extremely small compared with the whole universe, so we may regard space is flat enough there. Moreover, we may regard a change of the background scalar field ϕ_U is quasi-static. The perturbation of the local gravitional field is small enough, and the weak field approximation is available. Therefore we get the following gravitational field and the scalar field

$$\begin{cases} \phi=\phi_U(t)+2M_s/(3+2\eta)c^2r, \\ g_{00}=-1+(2M_s/\phi_U c^2r)[(4+2\eta)/(3+2\eta)], \\ g_{ii}=1+(2M_s/\phi_U c^2r)[(2+2\eta)/(3+2\eta)], \qquad i=1,2,3 \\ g_{\mu\nu}=0, \qquad \mu\neq\nu \end{cases} \tag{28}$$

where $M_s/\phi_U c^2 r \ll 1$, and M_s is mass of the Sun.

We write down the equation of motion (27) in the present field (28) at the rest-frame of the universe for non-special relativistic velocity, and get

$$\frac{3+2\eta}{4+2\eta}\cdot\phi_U\frac{dv}{dt}=-\frac{M_s}{r^2}, \tag{29}$$

where t exchanges to ct. In correspondence to the Newton theory, this equation means that we may interpret inertial mass m_I of the test-particle is determined as

$$m_I=\frac{3+2\eta}{4+2\eta}\cdot m_G\cdot G\phi_U(t) \tag{30}$$

for given gravitational mass m_G. Inertial mass is dominated by the cosmological scalar field ϕ_U. When matter exists in the universe

($\phi_U \neq 0$), the ratio m_I/m_G of inertial mass to gravitational mass is always same for all particles, and the principle of equivalence is satisfied. If both mass is represented by a same unit, the following relation must be satisfied at the present time:

$$\phi_U(t_0)=\frac{4+2\eta}{3+2\eta}\cdot\frac{1}{G}. \tag{31}$$

This relation prescribes the gravitational mass density of matter in the universe.

As the universe expands, matter attenuates in time, and the density of matter and the cosmological scalar field ϕ_U decrease to zero. Let us examine motion of a test-particle in empty space. In the limit $\rho\to 0$, inertial mass m_I vanishes, and the test-particle does not have inertia. As the external force does not act on the particle, the acceleration $d^2x^\mu/d\tau^2$ becomes indefinite. In empty space the acceleration itself loses its meaning. We understand it results from existing no reference frames (bodies) in empty space. When only one body exists in the universe and produce a finite gravitational force, the acceleration of the particle relative to that body diverges to infinity ($d^2x^\mu/d\tau^2\to\infty$) for the particle does not have inertia. However, in this case, the proper inertial force $-\phi_U d^2x^\mu/d\tau^2$ becomes finite and balances the gravitational force. As the apparent inertial force which appears in a reference frame accelerating to the rest-frame of the universe with a finite acceleration a is described as $-\phi_U a$, it converges to zero in the limit $\phi_U\to 0$. In empty space, the apparent inertial force does not appear.

The equation of motion in the Einstein theory correspondent to Eq. (29) is obviously

$$\frac{dv}{dt}=-\frac{GM_s}{r^2}. \tag{32}$$

In the Einstein theory the principle of equivalent is assumed and inertial mass m is given *a priori* to the test-particle. The particle has inertia even in empty space. Similarly the apparent inertial force $-ma$ appears irrespectively of existence of matter in the universe. Thus you would find that the material origin of inertia is extremely vague in the Einstein theory. This situation is almost same in the Brans-Dicke theory which has a constant scalar field ϕ_0.

(iv) inertial frame dragging

Now you know that a free-test-particle does not accelerate to the distribution of matter in the universe, and that when an external force acts on the particle, the proper inertial force appears and the particle resists against the over acceleration. Particles have a property to maintain the state of motion to matter of the universe. In another viewpoint, matter of the universe has a property to keep the proper inertial frame, which does not accelerate to the smoothed-out universe. We call this physical mechanism "*inertial frame dragging*".

Let us discuss inertial frame dragging arisen by a spherical rotating shell with the same density in the background universe[5]. At first the metric of the universe is given as Eq. (12). As the rotating shell becomes thicker, the perturbation of the metric tensor gradually becomes larger, and we obtain, as a result of the field equations,

$$ds^2 = -dt^2 + a^2(t)\{d\chi^2 + \sin^2\chi[d\theta^2 + \sin^2\theta(d\varphi - \omega dt/c)^2]\}, \qquad (33)$$

where calculations are restricted to the first order of the angular velocity. By writing down the equation of motion for this metric, we find the Coriolis force appears in the vicinity of the origin of the universe in the original reference frame. By the coordinate transformation $\varphi \to \varphi' = \varphi - \omega t/c$, we get

$$ds^2 = -dt^2 + a^2(t)\,[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi'^{\,2})], \tag{34}$$

and this means that the inertial frame at the origin rotates with the angular velocity ω to the original reference frame. We understand the rotating shell partially drags the inertial frame. When the shell covers the whole universe, the angular velocity ω of the inertial frame converges to that of the rotating shell ω_s, and the complete dragging is realized. Therefore the inertial frame is fixed to matter of the universe.

Thus the gravitational field obtained as a result of the inner construction of the field equations, inertial frame dragging, coincides with that by the mere coordinate transformation, and the general covariance as formality of the field equations is assured physically. This is a physical reason why relativity of motion is realized.

Even though the universe expands and matter attenuates, this situation is unchanged. However the density of matter in the universe decreases, the cosmological distribution of matter preserves the proper inertial frame as a whole. In the reality complete empty space does not exist.

You will find in the above discussion that Mach's original ideas are realized in the bucket.

V. Discussions and Summary

In the Einstein theory of gravitation, the equation of motion of a test-particle in a given gravitational field is that of geodesic in the correspondent Riemannian manifold. This geometrical interpretation may be enough to describe phenomena arizen in the gravitational field. It may not be necessary to question what the material origin of inertia is, and what the inertial force appears by the acceleration to. It may be enough that a state of nongravitation is surely realized in the elevator in Einstein's gedanken experiment. In the Brans-Dicke theory, it may not be necessary to question the physical meaning of the scalar field. It may be enough that we can find what gravitational field is produced as a whole, and how a test-particle moves in the gravitational field. However, sure enough, are you contented with that?

The Newton theory is sufficiently perfect to describe motion of solar system. However, within the framework of the theory, we can not understand physical meanings of absolute space, the global inertial frame, the inertial force, inertial mass, gravitational mass, the gravitational constant, and so on.

The Newton theory goes to ruin in the strong gravitational field, and there the Einstein theory relieves it. In the Einstein theory, the gravitational field is described geometrically by means of curved space-time. The global inertial frame loses its meaning, and the local inertial frame relieves it. All other physical laws are described in this reference frame. The metric tensor of the Riemannian manifold, therefore the local inertial frame seems to be determined by the distribution of matter through the field equations. However, as for this point, the interpretation is controversial. Because boundary conditions play an important role to determine a solution of the field equations.

In this theory the strong principle of equivalence is presumed *a priori*, and we can not understand a physical meaning of equivalence between the gravitational and the inertial forces or between gravitational and inertial mass. The gravitational constant is also assumed to be constant. In our viewpoint, ultimately, the Einstein theory is nothing but a local theory to describe the local gravitational field instantly at the present time in the evolution of our real universe. It is beyond the limits of the application of the theory to discuss a cosmological model as a whole.

In the Brans-Dicke theory the weak principle of equivalence is presumed, and the gravitational "constant" is no more constant in space and time. It is described by the scalar field, which is determined by the distribution of matter in the universe. Thus we can understand the meaning of the gravitational constant in the Newton or the Einstein theory. However, the origin of inertia remains vague.

Our theory of gravitation introduces a different viewpoint from ordinary theories, that is, the distinction between global and local observers. This theory stands on simpler foundations, and therefore becomes more comprehensive.

What is the inertial frame? It is a reference frame which rests to the smoothed-out distribution of matter in the universe. We call it the proper local inertial frame especially.

The inertial force is induced in a reference frame which accelerates relatively to the proper inertial frame, the rest-frame of the universe.

The essence of the inertial force is gravitational, and its material origin is sought to matter in the universe.

Inertial mass derives from the scalar interaction between gravitational mass of the particle and matter in the universe. The weak principle of equivalence is satisfied as a result of the theory.

The gravitational and the inertial forces are locally indistinguishable, and the local inertial frame can always be constructed in the gravitational field at any points.

When matter exists in the universe, the equation of motion of an infinitesimally small test-particle in the given external gravitational field coincides with that of geodesic in the Riemannian manifold. In the result, our theory describes same gravitational phenomena as those in the Brans-Dicke theory.

It makes the material origin of inertia clearer to discuss empty space. In the Einstein theory, even in empty space, inertia of a test-particle exists. Suppose Einstein's gedanken experiment in empty space. In the Brans-Dicke theory, too, inertia generally appears in empty space, as a matter-free scalar field ϕ_0 exists. In the present background cosmological model, the matter-free scalar field does not exist, and the cosmological scalar field ϕ_U converges to zero in the limit $\rho \rightarrow 0$. Therefore, in our theory, inertial mass vanishes and a particle does not have inertia in empty space. The particle receives an infinite acceleration when a finite force acts on it. The apparent inertial force which is induced in a reference frame accelerating to the rest-frame of the universe with a finite acceleration also vanishes. These results are formally obtained without the weak field approximation.

Strictly speaking, our theory itself is ruined when matter does not exist in the universe. The acceleration becomes indefinite and loses its meaning in empty space. The principle of inertia premises with the distribution of matter in the universe. We can not suppose perfectly empty space. So saying empty space, we actually introduce influences from the universe as external parameters, on the premise with the distribution of matter in the universe. Including-relations of the Newton theory, the Einstein theory, and the Brans-Dicke theory truly indicate that.

Now let us discuss a paradox. In explaining intuitively that space is curved when a gravitational field exists, we often take up a rotating disk in empty space. By the Lorentz contraction, the ratio of the circumference of the disk to its radius becomes smaller than 2π. After all, this means space is positively curved due to the gravitational field, for we may exchange the centrifugal force appearing in the rotating disk for the gravitational force according to the principle of equivalence. If the inertial force does not appear to the rotating frame in empty space, *what situation happens ?* In spite that space is surely curved by means of the rotation, the correspondent gravitational field does not exist. What does this mean ?

Moreover, the apparent inertial force itself is induced by a mere coordinate transformation in a meaning, and so it seems strange that it does not appear in empty space.

After all we conclude that a rotating frame like this does not exist in empty space. Generally speaking, we can not construct a reference frame with a finite acceleration in empty space. A reference frame is not completely arbitrary, therefore a coordinate system, either. A coordinate system to describe motion is the bodies in short, and only relative motion to them is meaningful. We can not consider a reference frame which can not be realized physically.

In the present theory the concept of the acceleration plays an essential role. We start from a fact that a free-particle in the smoothed-out universe does not accelerate to the distribution of matter in the universe, and obtain the equation of geodesic as that of motion in the gravitational field. In its derivation we assumed that the accelerating reference frame (elevator) is instantly rest to the rest-frame of the universe, but the results do not lose its generality as its formulation are covariant for the Lorentz transformation. The inertial force, in general relativistic theory, is also expressed as (inertial mass)×(acceleration) $\phi_U d^2\xi^\mu/d\tau^2$ as well as in the Newton theory, but $d^2\xi^\mu/d\tau^2$ is the four-acceleration of the reference frame to the distribution of matter in the universe.

In the special theory of relativity the equation of motion is given as

$$m\frac{d^2x^\mu}{d\tau^2}=f^\mu, \tag{35}$$

or

$$\frac{dp^\mu}{d\tau}=f^\mu, \tag{36}$$

where $p^\mu=mdx^\mu/d\tau$, and m is rest-mass, which is constant. In our theory inertial rest-mass varies in space and time and so these two equations are not equivalent to one another. From the above discuission we find that the equation (35) is rather more essential. The special theory of relativity is nothing but a local theory. We do not construct all physical laws in a gravitational field from special relativity through the principle of equivalence as in the Einstein theory. First of all the global theory of gravitation exists.

When a true external force f^μ acts on a test-particle in the gravitational field, the equation of motion is described as

$$\phi_U \frac{D^2 x^\mu}{D\tau^2} = f^\mu, \tag{37}$$

or

$$\phi_U \frac{d^2 x^\mu}{d\tau^2} = F^\mu + f^\mu, \tag{38}$$

where F^μ is the "gravitational" force given by Eq. (11) .This means the inertial force balances the external one in the rest-frame of the particle.

Inertial mass of a particle, in our theory, is determined by the scalar interaction between its gravitational mass and matter in the universe ($m_I \propto m_G \phi(x)$). Now we can examine Einstein's conjecture[9] : *The inertia of a body must increase when ponderable masses are piled up in its neighborhood.* A perturbation formula of the scalar field has already been solved[5]. According to this formula the contribution of an additional body to the scalar field at the origin of the universe is negative from the inside of χ_c, and positive from the outside. This means inertial mass of a particle decreases when a body is added in its neigyborhood, on the contrary to Einstein's conjecture. In closed space the sign of the contribution of the scalar field is reversed on the way. Inertial mass is almost dominated by distant matter in the universe and surely becomes positive as a whole.

A spatial change of inertial mass in the universe is extremely small, but a temporal change is not negligible. However, sure enough, how can we measure inertial mass itself independently ? According to Dirac's ideas[10] there are possibilities that other physical "constants" also vary in space and time.

The essence of our theory is found in recognition of roles of global and local observers. For local observers motion is completely relative. For global observers, though it is possible to say motion is relative to the last, the rest-frame of the universe plays a particular role and is the absolute reference frame to describe motion. The "*acceleration*" is, after all, the acceleration relative to the distribution of matter in the universe, and it induces the inertial force. As the acceleration loses its meaning in empty space, physically meaningful motion is a change of a position relative to other bodies in the universe. We can not so much as discusss motion in empty space which other bodies do not exist in.

Moreover, proper-time measured by a clock which rests to the rest-frame of the universe also has an absolute meaning. This cosmic time slices four-dimensional space-time universally, and is able to become an absolute parameter to describe motion.

Local observers can not creat a truly perfect theory. There is nothing for them to construct their theories only by means of local facts obtained from their direct experience. They can not know the meaning of parameters in their local theories. They merely determine their values by experiments. A global theory constructed by global observers essentially takes in all influences from the whole universe. They can understand the meaning of external parameters in local theories, and can determine their values cosmologically as a result of the theory, for our universe functions organically as a united whole.

In the past Aristotle's kinematics was cosmological. Newton excluded influences of the universe and created his system of dynamics by means of making all motion refer to absolute space representing its influences in the abstract.

Now, again mankind is laying his hands on the cosmological synthesis in physics.

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