Time-Variation of the Gravitational Constant and the Machian Solution in the Brans-Dicke Theory

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Abstract

We extend the Machian cosmological solution with the condition $\phi = O(\rho/\omega)$ to a perfect fluid with *negative* pressure by means of the method developed in the preceding paper and discuss some properties of the model. When the coefficient of the equation of state $\gamma \to -1/3$, the gravitational constant approaches to constant. If we assume the present mass density $\rho_0 \sim \rho_c$ (critical density), the parameter ϵ ($\epsilon \equiv 3\gamma + 1$) has a value of order 10^{-3} to support the present gravitational constant. The closed model is valid for $\omega < -3/2\epsilon$ and we understand why the coupling parameter $|\omega|$ is so large ($\omega \sim -10^3$). The time-variation of the gravitational constant $\left|\dot{G}/G\right| \sim 10^{-13}\, yr^{-1}$ at present is derived in this model.

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It has recently been cleared that the field equations of the Brans-Dicke theory [1] does not necessarily produce those of general relativity with the same energy-momentum tensor in the infinite limit of the coupling parameter ω [2], [3]. We have considered the physical essence of the difference between general relativity and the Brans-Dicke theory and have proposed the two postulates for the local and cosmological problems respectively [4]: *The scalar field by locally-distributed matter should show the asymptotic behavior* $\phi = \langle\phi\rangle + O(1/\omega)$ *for the large enough coupling parameter* ω, and *the scalar field of a proper cosmological solution should have the asymptotic form* $\phi = O(\rho/\omega)$ (the Machian solution).

We have systematically surveyed the general existence of such Machian cosmological solutions in the Brans-Dicke theory and have proved uniqueness of the Machian solution for the homogeneous and isotropic universe with a perfect fluid matter with negligible pressure [5]. However, it is unavoidable that the scalar field ϕ goes to zero as the universe expands satisfying the conservation law $a^3\rho = const$ in this cosmological model. The time-variation of the gravitational constant in this model is also not compatible with the recent observations (for examples [6], $\left|\dot{G}/G\right| \lesssim 1.6 \times 10^{-12}\, yr^{-1}$).

We will discuss some alternatives of matter to explain this experimental fact in the Machian point of view. First, we investigate simply the case of the vacuum energy. The mass density ρ_m of the perfect fluid with no pressure decreases gradually as the universe expands, and finally quantum corrections to the vacuum must not become negligible in matter. This vacuum energy density ρ_v must almost be constant even though the universe expands, and might keep the gravitational "constant" constant. Let us find the Machian solution with $\phi = O(\rho/\omega)$ for the vacuum energy in the Brans-Dicke theory.

The metric tensor for the homogeneous and isotropic universe is given as

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sigma^2(\chi)(d\theta^2 + \sin^2\theta d\varphi^2)]\,, \tag{1}$$

where

$$\sigma(\chi) \equiv \begin{cases} \sin\chi & for\ k = +1\ (closed\ space) \\ \chi & for\ k = 0\ \ (flat\ space) \\ \sinh\chi & for\ k = -1\ \ (open\ space)\,. \end{cases} \tag{2}$$

The source terms of the gravitational filed and the scalar field are the energy-momentum tensor of the perfect fluid with negligible pressure ($p = 0$) and the vacuum energy. The nonvanishing component of the energy-momentum tensor is $T_{00} = -\rho c^2$ and the contracted energy-momentum tensor is $T = \rho c^2$, where the total density $\rho = \rho_m + \rho_v$. Let us suppose that the mass density ρ_m obeys independently the conservation law $a^3\rho_m = const$ and the vacuum energy density ρ_v keeps constant. We discuss the epoch in which the relation $\rho_m \ll \rho_v$ is satisfied after the universe expands enough:

$$\rho = \rho_v = const\,. \tag{3}$$

The nonvanishing components of the field equations which we need solve simultaneously are

$$\frac{3}{a^2}\left(\dot{a}^2 + k\right) = \frac{16\pi(1+\omega)}{(3+2\omega)c^2}\frac{\rho}{\phi} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi} \tag{4}$$

and

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \frac{8\pi}{(3+2\omega)c^2}\rho\,, \tag{5}$$

where a dot denotes the partial derivative with respect to t.

We seek Machian solutions satisfying $\phi = O(\rho/\omega)$. Let us suppose that the scalar field ϕ is described as

$$\phi(t) = \frac{8\pi}{(3+2\omega)c^2}\Phi(t)\,, \tag{6}$$

where an unknown function $\Phi(t)$ depends on only t and should not include the coupling parameter ω in order that the scalar field $\phi(t)$ becomes Machian [5]. Substituting Eq.(6), we get from Eq.(5)

$$\ddot{\Phi} + 3\frac{\dot{a}}{a}\dot{\Phi} = \rho\,, \tag{7}$$

which means that the ratio $\dot{a}/a$ also includes only t as the vacuum energy density ρ_v does not depend on ω. So the expansion parameter $a(t)$ need have a form as

$$a(t) \equiv A(\omega)\alpha(t)\,, \tag{8}$$

where $\alpha(t)$ is a function of only t and a coefficient $A(\omega)$ includes only ω. Thus we obtain from Eq.(4) after eliminating $\ddot{\phi}$ by Eq.(5)

$$\frac{\omega}{2}\left[\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \frac{4\rho}{\Phi}\right] - \frac{3k}{A^2(\omega)\alpha^2} = 3\left(\frac{\dot{\alpha}}{\alpha}\right)^2 + 3\left(\frac{\dot{\alpha}}{\alpha}\right)\left(\frac{\dot{\Phi}}{\Phi}\right) - \frac{3\rho}{\Phi}\,. \tag{9}$$

For the closed and the open spaces ($k = \pm 1$), if we require that Eq.(9) is always satisfied for all arbitrary values of ω, we find the two following constraints must be held identically,

$$\frac{\omega}{2}\left[\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \frac{4\rho}{\Phi}\right] - \frac{3k}{A^2(\omega)\alpha^2} \equiv C(t) \tag{10}$$

and

$$3\left(\frac{\dot{\alpha}}{\alpha}\right)^2+3\left(\frac{\dot{\alpha}}{\alpha}\right)\left(\frac{\dot{\Phi}}{\Phi}\right)-\frac{3\rho}{\Phi}\equiv C(t)\,, \qquad (11)$$

where $C(t)$ is an arbitrary function of only t. From the constraint Eq.(10) for all arbitrary values of ω, we obtain that the coefficient $A(\omega)$ must have the following form

$$\frac{3}{A^2(\omega)}=\left|\frac{\omega}{2}+B\right|\,, \qquad (12)$$

where B is a constant with no dependence of ω.

Let us adopt a notation that $j=-1$ for $\omega/2+B<0$ and $j=+1$ for $\omega/2+B>0$, for simplicity. Taking this notation and Eq.(12) into account, we get from Eq.(9)

$$\frac{\omega}{2}\left[\left(\frac{\dot{\Phi}}{\Phi}\right)^2+\frac{4\rho}{\Phi}-k\,j\frac{1}{\alpha^2}\right]=3\left(\frac{\dot{\alpha}}{\alpha}\right)^2+3\left(\frac{\dot{\alpha}}{\alpha}\right)\left(\frac{\dot{\Phi}}{\Phi}\right)-\frac{3\rho}{\Phi}+k\,j\frac{B}{\alpha^2}\,. \qquad (13)$$

We need to hold the two following identities to satisfy this equation for all arbitrary ω:

$$\left(\frac{\dot{\Phi}}{\Phi}\right)^2+\frac{4\rho}{\Phi}\equiv k\,j\frac{1}{\alpha^2} \qquad (14)$$

and

$$3\left(\frac{\dot{\alpha}}{\alpha}\right)^2+3\left(\frac{\dot{\alpha}}{\alpha}\right)\left(\frac{\dot{\Phi}}{\Phi}\right)-\frac{3\rho}{\Phi}\equiv -k\,j\frac{B}{\alpha^2} \qquad (15)$$

for $k=\pm1$ and $j=\pm1$, respectively.

We have a prospect of existence of solutions which have the following form:

$$\Phi(t)=\zeta\rho(t)t^2\,, \qquad (16)$$

$$\alpha(t)=bt\,, \qquad (17)$$

where coefficients ζ and b are constants respectively. In fact, for $k\,j=+1$, we observe

$$\zeta=1/8\,,\quad b=1/6\,,\quad B=5/12 \qquad (18)$$

satisfies Eqs.(7), (14), and (15). We can determine ζ from Eq.(7), b from Eq.(14), and B from Eq.(15) successively. It should be noted that no Machian solutions exist for any combinations of $k\,j=-1$ in this case. If $\omega>-5/6$ for $k=+1$ or $-5/6>\omega>-2$ ($\omega\neq-3/2$) for $k=-1$, the gravitational force becomes attractive ($G>0$). Equation (16) gives the decreasing gravitational constant $G(t)\propto t^{-2}$ even if the vacuum energy supports the constant mass density ($\rho_v = const$) in this Machian solution. We can regard the vacuum energy as the cosmological constant Λ. If we observe the decaying cosmological constant $\Lambda(t) \propto t^{-2}$, this means the vacuum energy density $\rho_v(t) \propto t^{-2}$ and gives the constant scalar field $\Phi(t) = const$.

For the flat space case ($k = 0$), the two identities are derived from Eq.(13):

$$\left(\frac{\dot{\Phi}}{\Phi}\right)^2 + \frac{4\rho}{\Phi} \equiv 0\,, \tag{19}$$

and

$$\left(\frac{\dot{\alpha}}{\alpha}\right)^2 + \left(\frac{\dot{\alpha}}{\alpha}\right)\left(\frac{\dot{\Phi}}{\Phi}\right) - \frac{\rho}{\Phi} \equiv 0\,. \tag{20}$$

We find $\Phi(t) = -\rho_v t^2$ from Eq.(19) after integration (, taking the integral constant to zero) and $\Phi(t)\alpha^2(t) = const$ from Eq.(19) and (20), which gives $\alpha(t) \propto t^{-1}$. It is obvious that these functions $\Phi(t)$, $\alpha(t)$ do not satisfy Eq.(7) for $\rho_v = const$. No Machian solutions with the vacuum energy $\rho_v = const$ exist for the flat space.

Next we discuss Machian solutions for the perfect fluid with pressure p (see also [7]-[9]), the energy-momentum tensor of which is described as

$$T_{\mu\nu} = -pg_{\mu\nu} - (\rho + p/c^2)u_\mu u_\nu\,, \tag{21}$$

where u^μ is the four velocity $dx^\mu/d\tau$ (τ is the proper time). The nonvanishing components are $T_{00} = -\rho c^2$, $T_{ii} = -pg_{ii}$ ($i \neq 0$), and its trace is $T = \rho c^2 - 3p$ for the homogeneous and isotropic universe. The energy conservation $T^{\mu\nu}_{;\nu} = 0$ gives the equation of continuity

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p/c^2\right) = 0\,. \tag{22}$$

We suppose the equation of state

$$p(t) = \gamma\rho(t)c^2\,, \tag{23}$$

where $0 \leqq \gamma \leqq 1/3$ for the ordinary state. However, we will neglect this constraint here and consider the wider range of γ (at least, $-1 \leqq \gamma \leqq 1/3$). After integrating Eq.(22) with the equation of state, we obtain

$$\rho(t)\alpha^n(t) = const\,, \tag{24}$$

where $n = 3(\gamma + 1)$. Equations (4) and (5) change to

$$\frac{3}{a^2}\left(\dot{a}^2 + k\right) = \frac{16\pi(1+\omega)}{(3+2\omega)c^2}\frac{\rho}{\phi} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\ddot{\phi}}{\phi} + \frac{24\pi}{(3+2\omega)c^4}\frac{p}{\phi}\,, \tag{25}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \frac{8\pi}{(3+2\omega)c^2}\left(\rho - 3p/c^2\right) , \tag{26}$$

respectively in this case. Taking Eqs.(6) and (8) into account, we get directly after similar arguments

$$\ddot{\Phi} + 3\frac{\dot{\alpha}}{\alpha}\dot{\Phi} = \xi\rho \tag{27}$$

with $\gamma = (1-\xi)/3$ or $n = 4 - \xi$. After the elimination of $\ddot{\phi}$ from Eq.(25) using Eq.(26), we find that Eq.(13) holds for the perfect fluid with pressure as well as with no pressure.

For the flat space case ($k = 0$), we find the same identities Eqs.(19) and (20) from Eq.(13) for all ω, and then obtain

$$\Phi(t)\alpha^2(t) = const\,. \tag{28}$$

We have a prospect of existence of the following type of solution:

$$\Phi(t) = \zeta\rho(t)t^2\,, \tag{29}$$

$$\alpha(t) = bt^{\beta}\,, \tag{30}$$

where β is a constant. We observe from Eqs.(24),(27), and (28)

$$\zeta = 1/(\xi - 5)\,,\quad \beta = 2/(2-\xi)\,,\quad b : indefinite\,. \tag{31}$$

It is characteristic for the flat space that the coefficient b of the expansion parameter $a(t)$ becomes indefinite. No other Machian solutions exist in the range $0 \leqq \xi < 2$ for the flat space, because this solution is continuous in this region for the continuous parameter ξ including $\xi = 1$, for which the statement is proved [5] The coefficient ζ is negative for all ξ ($0 \leqq \xi \leqq 4$) or n ($4 \geqq n \geqq 0$), so the gravitational force becomes attractive for $\omega < -2$. The time-dependence of $\Phi(t)$ is described explicitly as

$$\Phi(t) \propto t^{-4/(\xi-2)}\,. \tag{32}$$

The sign of the power reverses at $\xi = 2$ or $n = 2$. There are no parameters to give a solution satisfying $\Phi(t) = const.$

For the closed and the open spaces ($k = \pm 1$), the equations which we need solve simultaneously are Eqs.(14), (15), (24), and (27). Similarly, we have a prospect of existence of the following type of solution:

$$\Phi(t) = \zeta\rho(t)t^2\,, \tag{33}$$

$$\alpha(t) = bt\,. \tag{34}$$

After calculating the power of $\Phi(t)$ in t by Eqs.(33), (34), and (24), we obtain

$$\zeta = 1/(\xi - 2) \tag{35}$$

from Eq.(27),

$$b = \begin{cases} (4-\xi^2)^{-1/2}, & for\ k\,j = -1\ and\ 0 \leqq \xi < 2 \\ (\xi^2 - 4)^{-1/2}, & for\ k\,j = +1\ and\ 2 < \xi \leqq 4 \end{cases} \tag{36}$$

from Eq.(14), and

$$B = -3/(\xi^2 - 4) \tag{37}$$

from Eq.(15) successively. Thus, the Machian solution also exists in these cases and unique for $0 \leqq \xi < 2$ because of continuity of the parameter ξ from $\xi = 1$ [5]. Though the parameter ξ is constant for this solution, the same solution holds if ξ varies slowly enough as the quasi-static process.

At the boundary $(\omega/2 + B = 0)$ between the closed and the open spaces, there exists a flat solution, which satisfies Eqs.(14), (15), (24), and (27) only for a particular value of the coupling constant $\omega = 6/(\xi^2 - 4)$. In this solution, the coefficient b becomes indefinite in the same way as the other cases for the flat space. The parameter $\xi = 1$ gives $\omega = -2$, which means $G = 0$. At $\xi = 0$ (for the universe with radiation), this solution becomes singular $(\omega = -3/2)$ and so we should discard it. We make the meaning of "uniqueness" [5] more definite by considering general cases for the perfect fluid with pressure.

The signs of the coefficient ζ and the parameter $k\,j$ reverse at $\xi = 2$ or $n = 2$ for the closed and the open spaces. To realize the attractive gravitational force $(G > 0)$, we restrict to $\omega < -2$ for $k = +1$ and $0 \leqq \xi \leqq 1$, to $\omega < 6/(\xi^2 - 4)$ for $k = +1$ and $1 < \xi < 2$, and to $6/(\xi^2 - 4) < \omega < -2$ for $k = -1$ and $1 < \xi < 2$. Moreover, when $2 < \xi \leqq 4$, we require $-2 < \omega < 6/(\xi^2 - 4)$ for $k = +1$ and $6/(\xi^2 - 4) < \omega$ for $k = -1$. In any cases, we exclude $\omega = -3/2$. The solution holds the type of $\Phi(t) \propto \rho(t)t^2$ and $a(t) \propto t$, and so the Machian relation

$$\frac{G(t)M}{c^2 a(t)} = const \tag{38}$$

is satisfied for all the time regardless of the time-dependence of the mass density ρ.

In this Machian solution, the scalar field $\Phi(t)$ keeps almost constant near $n = 2$. Let us suppose that the present universe is described as the case $n = 2 + \epsilon$ $(\epsilon \ll 1)$, and then we get for the time-dependence of $\Phi(t)$

$$\Phi(t) = -(1/\epsilon)\rho(t)t^2 \propto t^{-\epsilon}. \tag{39}$$

If we adopt $t_0/c \sim 10^{10}\,yr$ as the age of our universe and assume $\epsilon \sim 10^{-2}$, we find

$$\left|\dot{\Phi}(t)/\Phi(t)\right| \propto \epsilon/(t/c) \sim 10^{-12}\,yr^{-1}\,, \qquad (40)$$

which is compatible with the observational date for the time-variation of the gravitational constant [6].

The recent measurements [10] for the coupling parameter ω gives a severe restriction $|\omega| \gtrsim 10^3$. Taking $\omega \sim -10^3$, $\epsilon \sim 10^{-2}$, $t_0/c \sim 10^{10}\,yr$, and the present gravitational constant $G_0 = 6.67 \times 10^{-8}\,dyn.cm^2.g^{-1}$ into account, we can estimate the present mass density $\rho_0 \sim 10^{-28}\,g.cm^{-3}$ from Eqs.(6) and (39), which is ten times as large as the critical density $\rho_c \sim 10^{-29}\,g.cm^{-3}$. If we presume $\rho_0 \sim \rho_c$, we obtain $\epsilon \sim 10^{-3}$ for the same other parameters, and its value gives $\left|\dot{G}/G\right| \sim 10^{-13}\,yr^{-1}$ at present.

The scalar field ϕ has the asymptotic form $\phi = O(\rho/\omega)$ for the large coupling constant ω in any cases discussed here, and so the term $\omega(\dot{\phi}/\phi)^2$ appeared in the field equations does not vanish in the infinite limit of ω. The solution $\zeta = -1/2$, $b = 1/2$, and $B = 3/4$ for the closed or open universe with radiation ($\gamma = 1/3$, $T = 0$) does not show the asymptotic behavior $\phi = \langle\phi\rangle + O(1/\sqrt{\omega})$, though $T = 0$, which may be a counterexample to Faraoni [3]. The solution $\zeta = -1/5$, $\beta = 1$ for the flat universe with radiation is also another counterexample.

No Machian solutions for the flat space and for the closed or the open spaces with $k\,j = -1$ in the case of the vacuum energy $\rho_v = const.$ The sign of $k\,j$ for the solution with the vacuum energy $\rho_v = const$ is opposite to that of the solution with the mass density $\rho = const$ ($\xi = 4$, $n = 0$). It should be noted that there is a discontinuity at $\xi = 2$ and the sign of $k\,j$ ($\omega/2 + B > 0$ or < 0) reverses there. The flat solution for the perfect fluid with pressure does not satisfy $\Phi(t) = const.$ The closed solution in the range of $0 \leqq \xi < 2$ seems to be advantageous in the Machian point of view, taking the continuity of the parameter ξ from $\xi = 1$ and the sign of the gravitational constant. The solution with $\xi = 3$ or $\xi = 4$ is not continuously connected with that of $\xi = 1$ as the quasi-static process of ξ.

The parameter $\xi = 2$ which gives the Brans-Dicke scalar field $\phi(t) = const$ means $\gamma = -1/3$, that is a "negative" pressure. This may be mysterious, but recent measurements [11] of the distances to type Ia supernovae have revealed that the expansion of the universe is rather accelerating, which implies the existence of dark matter with negative pressure. A slow varying scalar field with negative pressure ($-1 < \gamma < 0$) is recently known as *quintessence* [12]. A cosmological constant behaves like matter with negative pressure $\gamma = -1$ [13]. So it does not seem to be necessarily unsound that we consider the existence of matter with negative pressure $\gamma = -1/3$.

The expansion parameter for the closed space ($0 \leqq \xi < 2$) is explicitly described as $a(t) = \left\{-6/\left[\left(4-\xi^2\right)\omega+6\right]\right\}^{1/2} t$ with the coupling parameter $\omega < -6/(4-\xi^2)$. For the case $n = 2+\epsilon$ ($\epsilon \sim 10^{-3}$), we get $a(t) \sim [-3/(2\epsilon\omega+3)]^{1/2}\, t$ with $\omega < -3/2\epsilon$ for the first order in ϵ. Thus, if we require $\omega = -3/\epsilon$, we find $a(t) \sim t$, and so we understand naturally the reason why the coupling parameter $|\omega|$ is so large ($\omega \sim -10^3$) at present. The expansion parameter $a(t)$ is a lincar function of t, but the parameter ξ approaches to 2 very slowly as the quasi-static process. Therefore, the expansion parameter increases a little faster than t for the constant coupling parameter ω and then the universe shows the slowly accelerating expansion.

Equations.(14), (15), (24), and (27) are invariant under the transformations $t \to t + t_c$, $t \to -t$, and $t \to t_c - t$ (t_c is a positive constant) for the solution Eqs.(33) and (34). So it is possible that this solution describes the expansion or the collapse from a finite radius with a finite gravitational constant.

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