Simple Derivation of the Coupling Function in the Generalized Scalar-Tensor Theory of Gravitation

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Abstract

In the Machian point of view, the more restrictive criterion $\phi \equiv -\Phi/\omega$ for the Brans-Dicke scalar field is introduced instead of $\phi = O(\rho/\omega)$ in the generalized scalar-tensor theory of gravitation. The scalar field ϕ should have the above form for all finite coupling parameter ω in cosmological considerations and the function Φ should not explicitly include ω. By means of this strong Machian criterion, the coupling function $\omega(\xi)$ is directly derived to be $\omega(\xi) = 3/(\xi - 2)$ without ambiguity. The Machian cosmological model is restricted to the closed universe due to this coupling function.

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In the previous paper entitled "Determination of the Coupling Function in the Generalized Scalar-Tensor Theory of Gravitation" [1], we discussed the cosmological behavior of the coupling function $\omega(\phi)$ of the generalized scalar-tensor theory of gravitation [2]-[4] in the framework of the Machian point of view, and proposed the extremely slow (quasi static) time-variable coupling function depending on the barotropic parameter ξ of matter in the universe, $\omega(\xi) = \omega_0/(\xi - 2)$.

The Machian criterion [5], [6], *the scalar field of a proper cosmological solution should have the asymptotic form* $\phi = O(\rho/\omega)$ *and should converge to zero in the continuous limit* $\rho/\omega \to 0$, restricts to a particular type (the Machian solution) as permissible cosmological solutions of the field equations for the homogeneous and isotropic universe with a perfect fluid [5] and [7], [8] in the Brans-Dicke theory of gravitation [9] or in the generalized scalar-tensor theory of gravitation. The Machian cosmological solution requests the severe constraint to a given constant coupling parameter ω, that is, $\omega/2 + B < 0$ ($B \equiv -3/(\xi + 2)(\xi - 2)$) in the region of $0 \leqq \xi < 2$ for the closed universe. This condition strongly suggests that the coupling parameter ω of the Brans-Dicke scalar field ϕ should vary in time as the barotropic evolution of matter in the universe and need behave like $\omega(t) \sim B(\xi)$, rather as a function of the barotropic parameter ξ, not the scalar field ϕ.

After several cosmological considerations of the Brans-Dicke scalar field, we attained to a possible determination of the coupling function, $\omega(\xi) = \omega_0/(\xi - 2)$ with $\omega_0 > 3$ (to avoid the singularity at $\omega = -3/2$). When the barotropic parameter ξ approaches to 2 (*negative pressure* $\gamma = -1/3$), the coupling function $\omega(\xi)$ diverges to minus infinity. This situation explains the reason why the coupling parameter $|\omega|$ is so large at present [10], $|\omega| \gtrsim 10^3$, if we regard the present barotropic state of the universe is $\epsilon \equiv 2 - \xi \sim 10^{-3}$, which is consistent with related observational data and their analysis [5], [11]. Time-variation of the gravitational constant vanishes and it dynamically converges to a finite value, which is also compatible with the recent observational data [12].

Recently, we discussed the stability of Machian cosmological models in the scalar-tensor theory of gravitation in comparison with the static Einstein universe in General Relativity and found the arbitrary constant ω_0 in the proposed coupling function should be fixed to 3 [13]. The Machian cosmological model, under this value $\omega_0 = 3$, always sustains the stability for the infinitesimal perturbation to the expansion parameter of the universe through the entire evolution except for the Big Bang, when the scalar-tensor theory of gravitation itself becomes singular. After this work was done, a simple and basic derivation of the coupling function $\omega(\xi)$ hits us. This analysis leads to the definite coupling function within the framework of the Machian point of view.

The action [2]-[4] for the generalized scalar-tensor theory of gravitation is described in our sign conventions ($c = 1$) as

$$S = \int d^4x\sqrt{-g}[-\phi R + 16\pi L_m - \frac{\omega(\phi)}{\phi} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}], \tag{1}$$

where R is the scalar curvature of the metric $g_{\mu\nu}$, $\phi(x)$ is the Brans-Dicke scalar field, $\omega(\phi)$ is an arbitrary coupling function, and L_m represents the Lagrangian for the matter fields. The variation of Eq.(1) with respect to $g_{\mu\nu}$ and ϕ leads to the field equations

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= \frac{8\pi}{\phi}T_{\mu\nu} - \frac{\omega(\phi)}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right) \\ &\quad -\frac{1}{\phi}(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi), \end{aligned} \tag{2}$$

$$\Box\phi = -\frac{1}{3 + 2\omega(\phi)}\left[8\pi T + \frac{d\omega(\phi)}{d\phi}\phi_{,\lambda}\phi^{,\lambda}\right], \tag{3}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, T is the contracted energy-momentum tensor, and $\Box$ denotes the generally-covariant d'Alembertian $\Box\phi \equiv \phi^{,\mu}{}_{;\mu}$.

We conceive that the coupling function is a function of the barotropic parameter ξ and so rewrite it as

$$\omega(\phi) \equiv \varpi(\xi), \tag{4}$$

and again we denote $\varpi(\xi)$ as $\omega(\xi)$ for simplicity. Our fundamental guiding principle is the Machian criterion, which restricts the Brans-Dicke scalar field ϕ to a particular type of function $\phi = O(\rho/\omega)$ for the large coupling parameter ω in the cosmological considerations. We try to make this postulate a little stronger and assert that the Brans-Dicke scalar field should have a following form even for the finite coupling parameter:

$$\phi \equiv -\Phi/\omega(\xi), \tag{5}$$

where a function Φ does not explicitly include ω. (Let us call this postulate the strong Machian criterion here.)

As the barotropic evolution of matter in the universe, the parameter ξ varies in time extremely slowly and thus the coupling function also varies quasi-statically. Therefore we may neglect the derivatives of $\omega(\xi)$ with respect to space-time x^μ for the homogeneous universe. After substituting Eq.(5) to Eq.(1) we obtain the new action for Φ

$$S' = \int d^4x\sqrt{-g}[-\Phi R - 16\pi\omega(\xi)L_m - \frac{\omega(\xi)}{\Phi}g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}]\,. \tag{6}$$

The third term of the action is invariant in its form for this transformation and the coupling function $\omega(\xi)$ remains the same. The Lagrangian for the matter fields is replaced to $-\omega(\xi)L_m$ and this term seems to imply that the coupling function is relevant to the matter fields. This conjecture will be set for future problems. From the action Eq.(6), considering $d\omega(\xi)/d\Phi = 0$, we similarly get the field equations

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= -\frac{8\pi\omega(\xi)}{\Phi}T_{\mu\nu} - \frac{\omega(\xi)}{\Phi^2}\left(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi_{,\lambda}\Phi^{,\lambda}\right) \\ &\quad -\frac{1}{\Phi}(\Phi_{,\mu;\nu} - g_{\mu\nu}\Box\Phi)\,, \end{aligned} \tag{7}$$

$$\Box\Phi = \frac{8\pi\omega(\xi)}{3+2\omega(\xi)}T\,. \tag{8}$$

The energy-momentum tensor (derived from L_m) for the perfect fluid with pressure p is given as

$$T_{\mu\nu} = -pg_{\mu\nu} - (\rho + p)u_\mu u_\nu\,, \tag{9}$$

where ρ is the mass density in comoving coordinates and u^μ is the four velocity $dx^\mu/d\tau$ (τ is the proper time). The nonvanishing components are $T_{00} = -\rho$, $T_{ii} = -pg_{ii}$ ($i \neq 0$), and its trace is $T = \rho - 3p$ for the homogeneous and isotropic universe. The barotropic equation of state is given as

$$p(t) = \gamma\rho(t)\,, \tag{10}$$

and the parameter γ is presumed to vary in the range of $-1/3 \leqq \gamma \leqq 1/3$, which corresponds with $0 \leqq \xi \leqq 2$, where the parameter ξ is defined by $\xi \equiv 1 - 3\gamma$, for the ordinary and negative pressure states. This extension of the range is relevant to unknown matter (dark-energy) suggested by the observed accelerating expansion of the universe [14].

Assuming the barotropic equation of state, we get the contracted energy-momentum tensor $T = \xi\rho$ and obtain directly

$$\Box\Phi = \frac{8\pi\omega(\xi)\xi}{3+2\omega(\xi)}\rho \tag{11}$$

from Eq.(8). The generally-covariant d'Alembertian is written down as $\Box\Phi \equiv -\ddot{\phi} + 3(\dot{a}/a)\dot{\phi}$ for the Robertson-Walker metric, and the ratio $\dot{a}/a$ does not include ω and ξ in our Machian cosmological models. Therefore, taking the strong Machian criterion into account, we conclude the coupling factor in Eq.(8) should not explicitly depend on ω and ξ, that is,

$$\frac{8\pi\omega(\xi)\xi}{3 + 2\omega(\xi)} \equiv C\,, \tag{12}$$

where C is an arbitrary constant.

From this constraint, we simply find the following relation:

$$\omega(\xi) = \frac{3}{8\pi\xi/C - 2}\,. \tag{13}$$

In the Machian cosmological model for the closed universe ($k = +1$), the coupling parameter ω is restricted to

$$\frac{\omega}{2} + B < 0\,, \tag{14}$$

where $B = -3/(\xi+2)(\xi-2)$, for a given barotropic parameter ξ. It is clear that the constant C in $\omega(\xi)$ must be $C = 8\pi$ for satisfying this inequality for all ξ $(0 \leqq \xi < 2)$ and thus we obtain as the coupling function

$$\omega(\xi) = \frac{3}{\xi - 2}\,. \tag{15}$$

Concerning the strong Machian criterion, if we adopt the relation $\phi \equiv -\Phi/[3 + 2\omega(\xi)]$ instead of Eq.(5), which is directly relevant to the Machian solution itself, we get $\Box\Phi = 8\pi\xi\rho$ as the correspondent equation to Eq.(11). The coupling function $\omega(\xi)$ vanishes completely in this equation, and thus $\omega(\xi)$ becomes indefinite. We can not determine the form of the coupling function within the framework of the Machian point of view. From the criterion $\phi \equiv -\Phi/\omega(\xi)$, we certainly reach the same Machian cosmological model because of the relation $1/[3 + 2\omega(\xi)] = 1/\omega(\xi)\xi$.

We may assume a more general relation, $\phi \equiv -\Phi/f(\omega)$, and can pursue the similar procedure. However, taking the criterion $\phi = O(\rho/\omega)$ for the large enough coupling parameter into account, we find the most general relation must be given by an arbitrary linear function $f(\omega) = A\omega + D$. For this modified strong Machian criterion, we get the general coupling function $\omega(\xi) = [(3 - (D/A)\xi])/(\xi - 2)$ with a pole at $\xi = 2$, where the ratio D/A is an arbitrary nonpositive constant. When $|\omega|$ is large enough, that is, in the vicinity of $\xi = 2$, the coupling function must behave like $\omega(\xi) = 3/(\xi - 2)$ according to the above result. Therefore we need fix the arbitrary constant D to 0. Though the constant A remains arbitrary, the correspondent Machian solution is invariant to a change of A. Hence, without generality, we may adopt the relation $\phi \equiv -\Phi/\omega(\xi)$ as our strong Machian criterion for the effective constraint to cosmological solutions.

The procedure to derive the coupling function $\omega(\xi)$ from the strong Machian criterion does not depend on a phase of geometry of the universe. Therefore the coupling function $\omega(\xi) = 3/(\xi - 2)$ is valid even for the flat or open model. However, the Machian cosmological model requests the condition $\omega/2 + B = 0$ or $\omega/2 + B > 0$ in the region of $0 \leqq \xi < 2$ for the flat or open universe respectively. This situation means that the Machian cosmological model excludes the flat and the open cases, which are merely permissible for a given coupling parameter ω, by means of cosmological considerations, the physical evolution of the barotropic parameter ξ and the coupling function $\omega(\xi)$.

We have derived the coupling function $\omega(\xi) = 3/(\xi - 2)$ from the strong Machian criterion first, and thus the stability of the Machian cosmological model for the entire evolution, too. The variable (*ad hoc*) cosmological constant $\lambda_1 \phi_{,\mu}\phi^{,\mu}/\phi^2$ may also be introduced to the action for explaining the observed slowly accelerating expansion of the universe. We can derive the similar Machian cosmological solution, by replacing formally $\omega(\xi)$ with $\omega(\xi) + \lambda_1$.

The remaining problem is why the coupling function directly depends on the barotropic parameter ξ. What are the origin and its physical meaning of the coupling function? For more general substance of the universe, the coupling function might change its form from the present result. However, whatever the universe may consist of, the barotropic equation of state must be fundamental. This equation includes and describes even the quintessence (negative pressure) or the cosmological constant ($\gamma = -1$). We may regard our result of the coupling function $\omega(\xi)$ holds good enough for general cases described by the similar equation of state.

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References

[1] A.Miyazaki, gr-qc/0102105, 2001.

[2] P.G.Bergmann, Int. J. Theor. Phys. 1, 25 (1968).

[3] R.V.Wagoner, Phys. Rev. D1, 3209 (1970).

[4] K.Nordtvedt, Astrophys. J. 161, 1059 (1970).

[5] Trends in Dark Matter Research, J.Val Blain(Ed.), Nova Science Publishers, Inc.(2005), p.141-p.189.

[6] A.Miyazaki, gr-qc/0012104, 2000.

[7] A.Miyazaki, gr-qc/0101112, 2001.

[8] A.Miyazaki, gr-qc/0102003, 2001.

[9] C.Brans and R.H.Dicke, Phys. Rev. 124, 925 (1961).

[10] X.Chen and M.Kamionkowski, Phys. Rev. D60, 104036 (1999).

[11] A.Miyazaki, gr-qc/0103013, 2001.

[12] D.B.Guenther, L.M.Krauss, and P.Demarque, Astrophys. J. 498, 871 (1998).

[13] A.Miyazaki, Nagasaki Prefectural Journal 40-2, 133 (2006).

[14] S.Perlmutter, M.S.Turner, and M.White, Phys. Rev. Lett. 83, 670 (1999).