The quantities derived from linear parallel displacements along an infinitesimal parallelogram

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Abstract

The author studies objects provided from two linear parallel displacements. One is a difference of two parallel vectors provided from linear parallel displacements along two one way courses of an infinitesimal parallelogram. One of the others is a difference between an initial vector and a vector provided from a linear parallel displacement going around an infinitesimal parallelogram. The author guesses these objects are quantity to evaluate "a curvature" in each point of a Finsler space. To prove them is a problem of the future.

Keywords and phrases : linear parallel displacement, infinitesimal parallelogram, parallel vector, Finsler geometry.

Introduction

In Riemannian geometry, it is well known that a difference between an initial vector and a parallel vector provided by a parallel displacement along an infinitesimal parallelogram is evaluated by curvature. The author wants to study a similar thing by using a linear parallel displacement of Finsler geometry. It is not only calculations but also to clarify what kind of information about "a curvature". However, it is not done yet in this paper.

In §2 and §3, the author calculates a difference of a parallel vector with two cases. In Case I, by using two one way courses and, in Case II, by using a loop along an infinitesimal parallelogram the differences are obtained, respectively. In Riemannian geometry, these two differences are the same but in Finsler geometry they are different(Theorem 3.1). The important points in calculations are we must do them on TM and pay attentions to the degree of approximations and the tensorial property.

Linear parallel displacement is stated in detail in [4],[5] and [6] by the author. The terminology and notations are referred to the books [2] and [3]. Here the author greatly appreciates very useful suggestions and kindness of Prof.T.Aikou.

1 A linear parallel displacement along a curve

First, we put terminology and notations used in this paper. Let M be an n-dimensional differentiable manifold and $F\Gamma = (N^i_j(x,y), F^i_{jr}(x,y), C^i_{jr}(x,y))$ Finsler connection(or the coefficients of a Finsler connection $F\Gamma$), and all of objects appeared in this paper (curves, vector fields, etc) are differentiable. In additions, indexes $a, b, c, \cdots, h, i, j, k, l, m, \cdots$ run on 1 to n. Further let TM be the tangent bundle of M.

Now, for a vector field on a curve c with a parameter t, we give a following definition of linear parallel displacements along c.

Definition 1.1 *For a curve $c = (c^i(t))$ $(a \leq t \leq b)$ and a vector field $v = (v^i(t))$ along c, if the equation*

$$\frac{dv^i}{dt} + v^j F^i_{jr}(c, \dot{c})\dot{c}^r = 0 \qquad (\dot{c}^r = \frac{dc^r}{dt}) \tag{1.1}$$

is satisfied, then v is called a parallel vector field along c, and we call the linear map $\Pi_c :\ v(a) \longrightarrow v(b)$ a linear parallel displacement along c.

Remark 1.1 *We can see that the differential equation* (1.1) *is linear with respect to a vector field v. Then a linear parallel displacement Π_c is regular, namely, one to one and on to, because of the uniqueness of the solution of the differential equation* (1.1).

We state the geometrical meaning of Definition 1.1. Let $\mathcal{H}$ be the collection of horizontal vectors at every point (x, y) on the tangent bundle TM, namely

$$\mathcal{H} = \bigcup_{(x,y)\in TM} \left\{ z^i \frac{\delta}{\delta x^i} \in T_{(x,y)}TM \ \Big|\ \frac{\delta}{\delta x^i} :\ \text{horizontal bases of } T_{(x,y)}TM \right\}. \tag{1.2}$$

Then $\mathcal{H}$ is a subbundle of the bundle TTM.
$\mathcal{H}$ has a local coordinate system $\{(x^i, y^i, z^i)\}$. This system have the coordinate transformation $(x^i, y^i, z^i) \longrightarrow (\bar{x}^a, \bar{y}^a, \bar{z}^a)$ attended with a coordinate transformation $(x^i) \longrightarrow (\bar{x}^a)$ of M, where

$$\begin{cases} \bar{x}^a = \bar{x}^a(x) \\ \bar{y}^a = y^j \dfrac{\partial \bar{x}^a}{\partial x^j} \\ \bar{z}^a = z^j \dfrac{\partial \bar{x}^a}{\partial x^j}. \end{cases} \tag{1.3}$$

If we put $F^i_j = z^r F^i_{rj}$, then we can take a differential operator with respect to x^i

$$\frac{\delta^H}{\delta x^i} = \frac{\partial}{\partial x^i} - N^r_i \frac{\partial}{\partial y^r} - F^r_i \frac{\partial}{\partial z^r}. \tag{1.4}$$

Now, for a curve $c = (c^i(t))$ on M and a vector field $v = (v^i(t))$ along c, we take the lift $\tilde{c} = (c^i, \dot{c}^i, v^i)$ to $\mathcal{H}$ and calculate the tangent vector $\dfrac{d\tilde{c}}{dt}$ of $\tilde{c}$. Then we have

$$\begin{aligned} \frac{d\tilde{c}}{dt} &= \frac{dc^i}{dt}\frac{\partial}{\partial x^i} + \frac{d\dot{c}^i}{dt}\frac{\partial}{\partial y^i} + \frac{dv^i}{dt}\frac{\partial}{\partial z^i} \\ &= \dot{c}^i \frac{\delta^H}{\delta x^i} + \left(\frac{d\dot{c}^i}{dt} + N^i_r(c,\dot{c})\dot{c}^r\right)\frac{\partial}{\partial y^i} + \left(\frac{dv^i}{dt} + v^j F^i_{jr}(c,\dot{c})\dot{c}^r\right)\frac{\partial}{\partial z^i}. \end{aligned} \tag{1.5}$$

Therefore Definition 1.1 means that the lift $\tilde{c}$ is horizontal of $\mathcal{H}$. So we have

Proposition 1.1 *If a vector field $v = (v^i(t))$ along a curve $c = (c^i(t))$ is parallel along c, then the lift $\tilde{c} = (c, \dot{c}, v)$ to $\mathcal{H}$ is horizontal. The inverse property is also true.*

Remark 1.2 *Now, we already know a definition of a parallel displacement along c. It is as follows*

Definition 1.2 ([1]) *For a curve $c = (c^i(t))$ and a vector field $v = (v^i(t))$ along c, if the equations*

$$\frac{dv^i}{dt} + N^i_r(c, v)\dot{c}^r = 0 \qquad (\dot{c}^r = \frac{dc^r}{dt}) \tag{1.6}$$

are satisfied, then v is called the parallel vector field along the curve c.

This parallel displacement is not linear with respect to the vector field v and the meaning of this definition is for the lift $\tilde{c} = (c, v)$ to TM to be a horizontal curve on TM.

2 Linear parallel displacements along an infinitesimal parallelogram

We study two cases. One is the case that makes an initial vector be two parallel vector fields along two routes(Case I), and the other is the case making a parallel vector field along one loop(Case II). Hereafter, we assume all points and curves are in one coordinate neighborhood and $F\Gamma$ satisfies the torsion tensor field $T^i_{jk}(x, y) = F^i_{jk}(x, y) - F^i_{kj}(x, y) = 0$. Further the coefficients $N^i_j(x, y), F^i_{jk}(x, y)$ are positively homogeneous degree 1 and 0 with respect to y, respectively.

Case I. Let p, q, r, s be four points on M and $(x^i), (x^i + \xi^i), (x^i + \xi^i + \eta^i), (x^i + \eta^i)$ coordinates, respectively. Further,c_1, c_2, c_3, c_4 are following curves with a parameter t ($0 \leq t \leq 1$):

$$(I) \begin{cases} c_1(t): & x^i(t) = x^i + t\xi^i \ (p \text{ to } q), \\ c_2(t): & x^i(t) = x^i + \xi^i + t\eta^i \ (q \text{ to } r), \\ c_3(t): & x^i(t) = x^i + t\eta^i \ (p \text{ to } s), \\ c_4(t): & x^i(t) = x^i + \eta^i + t\xi^i \ (s \text{ to } r). \end{cases}$$

We take two routes $c = c_1 + c_2 (p \to q \to r)$ and $\bar{c} = c_3 + c_4 (p \to s \to r)$, and consider linear parallel displacements along c and $\bar{c}$, respectively. Let $V = (V^i)$ be an initial vector at p and V_q, V_r the value at q and r by the parallel vector field along c, respectively. Further, let $\bar{V}_s, \bar{V}_r$ the value at s and r by the parallel vector field along $\bar{c}$, respectively(See Figure 1).

Our standpoint is to investigate the difference $\bar{V}_r - V_r$. First, we move the initial vector V from $p(x^i)$ to $q(x^i + \xi^i)$ by the linear parallel displacement along c_1 and obtain V_q. Then $V_q = V + dV$, $\frac{dV^i}{dt} + F^i_{hj}(x, \xi)V^h\xi^j = 0$ are satisfied. Therefore, we have

$$V^i_q = V^i - F^i_{hj}(x, \xi)V^h\xi^j. \tag{2.1}$$

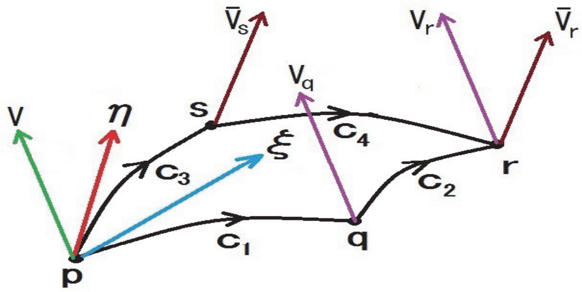

Figure 1: Case I

Next, we move V_q from $q(x^i+\xi^i)$ to $r(x^i+\xi^i+\eta^i)$ along c_2. Then $V_r = V_q + dV_q$, $\frac{dV_q^i}{dt} + F^i_{hj}(x+\xi,\eta)V_q^h\eta^j = 0$ are satisfied. In addition we have the following Taylor expansion

$$(2.2) \qquad F^i_{hj}(x+\xi,\eta) = F^i_{hj}(x,\xi) + \frac{\partial F^i_{hj}}{\partial x^k}(x,\xi)\xi^k + \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(\eta^k-\xi^k) + \cdots .$$

So we have

(2.3)

$$\begin{aligned} V_r^i &= V_q^i - F^i_{hj}(x+\xi,\eta)V_q^h\eta^j \\ &= V^i - F^i_{hj}(x,\xi)V^h\xi^j \\ &\quad - (F^i_{hj}(x,\xi) + \frac{\partial F^i_{hj}}{\partial x^k}(x,\xi)\xi^k + \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(\eta^k-\xi^k) + \cdots)(V^h - F^h_{ml}(x,\xi)V^m\xi^l)\eta^j \\ &= V^i - F^i_{hj}(x,\xi)V^h\xi^j - F^i_{hj}(x,\xi)V^h\eta^j \\ &\quad - \frac{\partial F^i_{hj}}{\partial x^k}(x,\xi)\xi^kV^h\eta^j - \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(\eta^k-\xi^k)V^h\eta^j + F^i_{hj}(x,\xi)F^h_{ml}(x,\xi)V^m\xi^l\eta^j + \cdots . \end{aligned}$$

Further, by the similar calculation for $\bar{V}_r$, we have

(2.4)

$$\begin{aligned} \bar{V}_r^i &= V^i - F^i_{hj}(x,\eta)V^h\eta^j - F^i_{hj}(x,\eta)V^h\xi^j \\ &\quad - \frac{\partial F^i_{hj}}{\partial x^k}(x,\eta)\eta^kV^h\xi^j - \frac{\partial F^i_{hj}}{\partial y^k}(x,\eta)(\xi^k-\eta^k)V^h\xi^j + F^i_{hj}(x,\eta)F^h_{ml}(x,\eta)V^m\eta^l\xi^j + \cdots , \end{aligned}$$

where we use the following Taylor expansion at s

$$(2.5) \qquad F^i_{hj}(x+\eta,\xi) = F^i_{hj}(x,\eta) + \frac{\partial F^i_{hj}}{\partial x^k}(x,\eta)\eta^k + \frac{\partial F^i_{hj}}{\partial y^k}(x,\eta)(\xi^k-\eta^k) + \cdots .$$

Remark 2.1 *In* (2.3) *and* (2.4), $(\cdots)$ *expresses 3rd and more order terms with respect to* ξ,η. *We use this expression in this paper, frequently.*

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Therefore the following equation, from (2.3) and (2.4)

(2.6)

$$
\begin{aligned}
&\bar{V}^i_r - V^i_r = F^i_{hj}(x,\xi)V^h\xi^j + F^i_{hj}(x,\xi)V^h\eta^j \\
&+ \frac{\partial F^i_{hj}}{\partial x^k}(x,\xi)\xi^k V^h\eta^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(\eta^k-\xi^k)V^h\eta^j - F^i_{hj}(x,\xi)F^h_{ml}(x,\xi)V^m\xi^l\eta^j \\
&- F^i_{hj}(x,\eta)V^h\eta^j - F^i_{hj}(x,\eta)V^h\xi^j \\
&- \frac{\partial F^i_{hj}}{\partial x^k}(x,\eta)\eta^k V^h\xi^j - \frac{\partial F^i_{hj}}{\partial y^k}(x,\eta)(\xi^k-\eta^k)V^h\xi^j + F^i_{hj}(x,\eta)F^h_{ml}(x,\eta)V^m\eta^l\xi^j + \cdots \\
&= F^i_{hj}(x,\xi)V^h\xi^j + F^i_{hj}(x,\xi)V^h\eta^j - F^i_{hj}(x,\eta)V^h\eta^j - F^i_{hj}(x,\eta)V^h\xi^j \\
&+ \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(\eta^k-\xi^k)V^h\eta^j - \frac{\partial F^i_{hj}}{\partial y^k}(x,\eta)(\xi^k-\eta^k)V^h\xi^j \\
&+ \frac{\partial F^i_{hj}}{\partial x^k}(x,\xi)\xi^k V^h\eta^j - \frac{\partial F^i_{hj}}{\partial x^k}(x,\eta)\eta^k V^h\xi^j - F^i_{hj}(x,\xi)F^h_{ml}(x,\xi)V^m\xi^l\eta^j + F^i_{hj}(x,\eta)F^h_{ml}(x,\eta)V^m\eta^l\xi^j + \cdots \\
&= [(F^i_{hj}(x,\xi) - F^i_{hj}(x,\eta))(\xi^j+\eta^j) + (\frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)\eta^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\eta)\xi^j)(\eta^k-\xi^k) \\
&+ (\frac{\partial F^i_{hj}}{\partial x^k}(x,\xi) - \frac{\partial F^i_{hk}}{\partial x^j}(x,\eta) - F^i_{mj}(x,\xi)F^m_{hk}(x,\xi) + F^i_{mk}(x,\eta)F^m_{hj}(x,\eta))\eta^j\xi^k]V^h + \cdots
\end{aligned}
$$

is satisfied. So we have

Proposition 2.1 *Let M be an n-dimensional differentiable manifold with a Finsler connection $F\Gamma = (N^i_j(x,y), F^i_{jk}(x,y), C^i_{jk}(x,y))$ satisfying $T^i_{jk}(x,y) = 0$. For an infinitesimal parallelogram defined by (I) and an initial vector $V = (V^i)$, we have vectors $V_r = (V^i_r), \bar{V}_r = (\bar{V}^i_r)$. Then the difference $\bar{V}^i_r - V^i_r$ is written in the following form*

(2.7)

$$
\begin{aligned}
\bar{V}^i_r - V^i_r &= [(F^i_{hj}(x,\xi) - F^i_{hj}(x,\eta))(\xi^j+\eta^j) + (\frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)\eta^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\eta)\xi^j)(\eta^k-\xi^k) \\
&+ (\frac{\partial F^i_{hj}}{\partial x^k}(x,\xi) - \frac{\partial F^i_{hk}}{\partial x^j}(x,\eta) - F^i_{mj}(x,\xi)F^m_{hk}(x,\xi) + F^i_{mk}(x,\eta)F^m_{hj}(x,\eta))\eta^j\xi^k]V^h + \cdots .
\end{aligned}
$$

Case II. Let four points p, q, r, s be the same in Case I. However, curves c_3, c_4 are different from (I) as follows

$$
(II)\begin{cases}
c_1(t): & x^i(t) = x^i + t\xi^i \ (p \text{ to } q), \\
c_2(t): & x^i(t) = x^i + \xi^i + t\eta^i \ (q \text{ to } r), \\
c_3(t): & x^i(t) = x^i + \xi^i + \eta^i - t\xi^i \ (r \text{ to } s), \\
c_4(t): & x^i(t) = x^i + \eta - t\eta^i \ (s \text{ to } p),
\end{cases}
$$

where $0 \le t \le 1$.

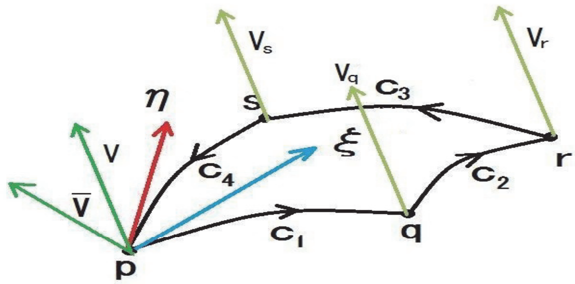

Figure 2: Case II

the parallel vector field along c at q, r, s, respectively. Further, let $\bar{V}$ be the value at the end point p(See Figure 2).

In this time, we investigate the difference $\bar{V} - V$. First, we have the same formation (2.3) as V_r until r. Next, from $V_s = V_r + dV_r$, $\frac{dV_r^i}{dt} + F_{hj}^i(x+\xi+\eta, -\xi)V_r^h(-\xi^j) = 0$, we have

$$(2.8)\quad \begin{aligned} V_s^i = V^i &- F_{hj}^i(x,\xi)V^h\xi^j - F_{hj}^i(x,\xi)V^h\eta^j \\ &- \frac{\partial F_{hj}^i}{\partial x^k}(x,\xi)\xi^k V^h\eta^j - \frac{\partial F_{hj}^i}{\partial y^k}(x,\xi)(\eta^k - \xi^k)V^h\eta^j + F_{hj}^i(x,\xi)F_{ml}^h(x,\xi)V^m\xi^l\eta^j \\ &- F_{hj}^i(x+\xi+\eta,-\xi)V_r^h(-\xi^j) + \cdots . \end{aligned}$$

In addition, the following Taylor expansion

$$(2.9)\quad F_{hj}^i(x+\xi+\eta,-\xi) = F_{hj}^i(x,-\eta) + \frac{\partial F_{hj}^i}{\partial x^k}(x,-\eta)(\xi^k+\eta^k) + \frac{\partial F_{hj}^i}{\partial y^k}(x,-\eta)(\eta^k-\xi^k) + \cdots$$

is satisfied. So we have

(2.10)

$$\begin{aligned} F_{hj}^i(x+\xi+\eta,-\xi)V_r^h\xi^j &= (F_{hj}^i(x,-\eta) + \frac{\partial F_{hj}^i}{\partial x^k}(x,-\eta)(\xi^k+\eta^k) + \frac{\partial F_{hj}^i}{\partial y^k}(x,-\eta)(\eta^k-\xi^k) + \cdots)\times \\ &\times (V^h - F_{mb}^h(x,\xi)V^m\xi^b - F_{mb}^h(x,\xi)V^m\eta^b - \frac{\partial F_{mb}^h}{\partial x^k}(x,\xi)\xi^k V^m\eta^b \\ &\qquad - \frac{\partial F_{mb}^h}{\partial y^k}(x,\xi)(\eta^k-\xi^k)V^m\eta^b + F_{ab}^h(x,\xi)F_{ml}^a(x,\xi)V^m\xi^l\eta^b + \cdots)\xi^j \\ &= F_{hj}^i(x,-\eta)V^h\xi^j - F_{hj}^i(x,-\eta)F_{mb}^h(x,\xi)V^m\xi^b\xi^j - F_{hj}^i(x,-\eta)F_{mb}^h(x,\xi)V^m\eta^b\xi^j \\ &\qquad + \frac{\partial F_{hj}^i}{\partial x^k}(x,-\eta)(\xi^k+\eta^k)V^h\xi^j + \frac{\partial F_{hj}^i}{\partial y^k}(x,-\eta)(\eta^k-\xi^k)V^h\xi^j + \cdots . \end{aligned}$$

From (2.8) and (2.10),

(2.11)
$$\begin{aligned}V_s^i = V^i &- F_{hj}^i(x,\xi)V^h\xi^j - F_{hj}^i(x,\xi)V^h\eta^j\\ &- \frac{\partial F_{hj}^i}{\partial x^k}(x,\xi)\xi^k V^h\eta^j - \frac{\partial F_{hj}^i}{\partial y^k}(x,\xi)(\eta^k-\xi^k)V^h\eta^j + F_{hj}^i(x,\xi)F_{ml}^h(x,\xi)V^m\xi^l\eta^j\\ &+ F_{hj}^i(x,-\eta)V^h\xi^j - F_{hj}^i(x,-\eta)F_{mb}^h(x,\xi)V^m\xi^b\xi^j - F_{hj}^i(x,-\eta)F_{mb}^h(x,\xi)V^m\eta^b\xi^j\\ &+ \frac{\partial F_{hj}^i}{\partial x^k}(x,-\eta)(\xi^k+\eta^k)V^h\xi^j + \frac{\partial F_{hj}^i}{\partial y^k}(x,-\eta)(\eta^k-\xi^k)V^h\xi^j + \cdots\end{aligned}$$

is satisfied.

Finally, from $\bar{V} = V_s + dV_s$, $\frac{dV_s^i}{dt} + F_{hj}^i(x+\eta,-\eta)V_s^h(-\eta^j) = 0$,

(2.12)
$$\begin{aligned}\bar{V}^i = V^i &- F_{hj}^i(x,\xi)V^h\xi^j - F_{hj}^i(x,\xi)V^h\eta^j\\ &- \frac{\partial F_{hj}^i}{\partial x^k}(x,\xi)\xi^k V^h\eta^j - \frac{\partial F_{hj}^i}{\partial y^k}(x,\xi)(\eta^k-\xi^k)V^h\eta^j + F_{hj}^i(x,\xi)F_{ml}^h(x,\xi)V^m\xi^l\eta^j\\ &+ F_{hj}^i(x,-\eta)V^h\xi^j - F_{hj}^i(x,-\eta)F_{mb}^h(x,\xi)V^m\xi^b\xi^j - F_{hj}^i(x,-\eta)F_{mb}^h(x,\xi)V^m\eta^b\xi^j\\ &+ \frac{\partial F_{hj}^i}{\partial x^k}(x,-\eta)(\xi^k+\eta^k)V^h\xi^j + \frac{\partial F_{hj}^i}{\partial y^k}(x,-\eta)(\eta^k-\xi^k)V^h\xi^j\\ &- F_{hj}^i(x+\eta,-\eta)V_s^h(-\eta^j) + \cdots\end{aligned}$$

is satisfied. In addition

$$F_{hj}^i(x+\eta,-\eta) = F_{hj}^i(x,-\eta) + \frac{\partial F_{hj}^i}{\partial x^k}(x,-\eta)\eta^k + \cdots \tag{2.13}$$

is also satisfied. Then we have

(2.14)
$$\begin{aligned}&F_{hj}^i(x+\eta,-\eta)V_s^h\eta^j = (F_{hj}^i(x,-\eta) + \frac{\partial F_{hj}^i}{\partial x^k}(x,-\eta)\eta^k + \cdots)(V^h - F_{ac}^h(x,\xi)V^a\xi^c - F_{ac}^h(x,\xi)V^a\eta^c\\ &- \frac{\partial F_{ac}^h}{\partial x^k}(x,\xi)\xi^k V^a\eta^c - \frac{\partial F_{ac}^h}{\partial y^k}(x,\xi)(\eta^k-\xi^k)V^a\eta^c + F_{ac}^h(x,\xi)F_{ml}^a(x,\xi)V^m\xi^l\eta^c\\ &+ F_{ac}^h(x,-\eta)V^a\xi^c - F_{ac}^h(x,-\eta)F_{mb}^a(x,\xi)V^m\xi^b\xi^c - F_{ac}^h(x,-\eta)F_{mb}^a(x,\xi)V^m\eta^b\xi^c\\ &+ \frac{\partial F_{ac}^h}{\partial x^k}(x,-\eta)(\xi^k+\eta^k)V^a\xi^c + \frac{\partial F_{ac}^h}{\partial y^k}(x,-\eta)(\eta^k-\xi^k)V^a\xi^c + \cdots)\eta^j\\ &= F_{hj}^i(x,-\eta)V^h\eta^j - F_{hj}^i(x,-\eta)F_{ac}^h(x,\xi)V^a\xi^c\eta^j - F_{hj}^i(x,-\eta)F_{ac}^h(x,\xi)V^a\eta^c\eta^j\\ &+ F_{hj}^i(x,-\eta)F_{ac}^h(x,-\eta)V^a\xi^c\eta^j + \frac{\partial F_{hj}^i}{\partial x^k}(x,-\eta)\eta^k V^h\eta^j + \cdots .\end{aligned}$$

From (2.12) and (2.14),

(2.15)
$$
\begin{aligned}
\bar{V}^i = V^i &- F^i_{hj}(x,\xi)V^h\xi^j - F^i_{hj}(x,\xi)V^h\eta^j \\
&- \frac{\partial F^i_{hj}}{\partial x^k}(x,\xi)\xi^k V^h\eta^j - \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(\eta^k-\xi^k)V^h\eta^j + F^i_{hj}(x,\xi)F^h_{ml}(x,\xi)V^m\xi^l\eta^j \\
&+ F^i_{hj}(x,-\eta)V^h\xi^j - F^i_{hj}(x,-\eta)F^h_{mb}(x,\xi)V^m\xi^b\xi^j - F^i_{hj}(x,-\eta)F^h_{mb}(x,\xi)V^m\eta^b\xi^j \\
&+ \frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)(\xi^k+\eta^k)V^h\xi^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,-\eta)(\eta^k-\xi^k)V^h\xi^j \\
&+ F^i_{hj}(x,-\eta)V^h\eta^j - F^i_{hj}(x,-\eta)F^h_{ac}(x,\xi)V^a\xi^c\eta^j - F^i_{hj}(x,-\eta)F^h_{ac}(x,\xi)V^a\eta^c\eta^j \\
&+ F^i_{hj}(x,-\eta)F^h_{ac}(x,-\eta)V^a\xi^c\eta^j + \frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\eta^k V^h\eta^j + \cdots .
\end{aligned}
$$

We can arrange (2.15) as follows

(2.16)
$$
\begin{aligned}
&\bar{V}^i = V^i - F^i_{hj}(x,\xi)V^h\xi^j - F^i_{hj}(x,\xi)V^h\eta^j + F^i_{hj}(x,-\eta)V^h\xi^j + F^i_{hj}(x,-\eta)V^h\eta^j \\
&+ \frac{\partial F^i_{hj}}{\partial y^k}(x,-\eta)(\eta^k-\xi^k)V^h\xi^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(\eta^k-\xi^k)V^h(-\eta^j) \\
&+ \frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\eta^k V^h\xi^j - \frac{\partial F^i_{hj}}{\partial x^k}(x,\xi)\xi^k V^h\eta^j + F^i_{hj}(x,\xi)F^h_{ml}(x,\xi)V^m\xi^l\eta^j - F^i_{hj}(x,-\eta)F^h_{ac}(x,-\eta)V^a\xi^j\eta^c \\
&- F^i_{hj}(x,-\eta)F^h_{mb}(x,\xi)V^m\xi^b\xi^j - F^i_{hj}(x,-\eta)F^h_{mb}(x,\xi)V^m\eta^b\xi^j \\
&- F^i_{hj}(x,-\eta)F^h_{mb}(x,\xi)V^m\xi^b\eta^j - F^i_{hj}(x,-\eta)F^h_{mb}(x,\xi)V^m\eta^b\eta^j \\
&+ \frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\xi^k V^h\xi^j + \frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\eta^k V^h\eta^j \\
&+ F^i_{hj}(x,-\eta)F^h_{ac}(x,-\eta)V^a\xi^j\eta^c + F^i_{hj}(x,-\eta)F^h_{ac}(x,-\eta)V^a\xi^c\eta^j + \cdots .
\end{aligned}
$$

Proposition 2.2 *Let M be an n-dimensional differentiable manifold with a Finsler connection $F\Gamma = (N^i_j(x,y), F^i_{jk}(x,y), C^i_{jk}(x,y))$ satisfying $T^i_{jk}(x,y) = 0$. For an infinitesimal parallelogram defined by (II) and an initial vector $V = (V^i)$, the parallel vector $\bar{V} = (\bar{V}^i)$ is obtained from a linear parallel displacement along c. Then $\bar{V}$ and V have the relation* (2.16).

Now the following equation

$$
\begin{aligned}
&F^i_{hj}(x,-\eta)F^h_{mb}(x,\xi)V^m\xi^b\xi^j + F^i_{hj}(x,-\eta)F^h_{mb}(x,\xi)V^m\eta^b\xi^j \\
&+ F^i_{hj}(x,-\eta)F^h_{mb}(x,\xi)V^m\xi^b\eta^j + F^i_{hj}(x,-\eta)F^h_{mb}(x,\xi)V^m\eta^b\eta^j \\
&= F^i_{hj}(x,-\eta)F^h_{mb}(x,\xi)V^m(\xi^b+\eta^b)(\xi^j+\eta^j)
\end{aligned}
\tag{2.17}
$$

is true. Further we allow the following equations

$$
\frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\xi^k\xi^j = F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)\xi^j\xi^k, \tag{2.18}
$$

$$
\frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\eta^k\eta^j = F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)\eta^j\eta^k. \tag{2.19}
$$

We can calculate the second order differential with respect to t because $\frac{dV_r^i}{dt} + F^i_{hj}(x+\xi+\eta,-\xi)V_r^h(-\xi^j) = 0$ is satisfied on the way from r to s.

$$(2.20)\quad \frac{d^2V_r^i}{dt^2} + (\frac{\partial F^i_{hj}}{\partial x^k}(x+\xi+\eta,-\xi)\xi^k\xi^j - F^i_{mj}(x+\xi+\eta,-\xi)F^m_{hk}(x+\xi+\eta,-\xi)\xi^k\xi^j)V_r^h = 0$$

We can ignore the second order differential, namely, $\frac{d^2V_r^i}{dt^2} = 0$, then we have

$$(2.21)\qquad \frac{\partial F^i_{hj}}{\partial x^k}(x+\xi+\eta,-\xi)\xi^k\xi^j - F^i_{mj}(x+\xi+\eta,-\xi)F^m_{hk}(x+\xi+\eta,-\xi)\xi^k\xi^j = 0.$$

From this equation and (2.9), if we ignore the terms of 3rd and more order with respect to ξ and η in it, we have (2.18). Further (2.19) is obtained by the same calculation from $\frac{dV_s^i}{dt} + F^i_{hj}(x+\eta,-\eta)V_s^h(-\eta^j) = 0$ by using (2.13).

Now from (2.18) and (2.19), the following equations is satisfied.

$$(2.22)\qquad \begin{aligned} &\frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\xi^k\xi^j + \frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\eta^k\eta^j \\ &\qquad + F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)\xi^j\eta^k + F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)\xi^k\eta^j \\ &= F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)\xi^j\xi^k + F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)\eta^j\eta^k \\ &\qquad + F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)\xi^j\eta^k + F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)\xi^k\eta^j \end{aligned}$$

Namely, we have

$$(2.23)\qquad \begin{aligned} &\frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\xi^k\xi^j + \frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\eta^k\eta^j \\ &\qquad + F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)\xi^j\eta^k + F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)\xi^k\eta^j \\ &= F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)(\xi^j+\eta^j)(\xi^k+\eta^k). \end{aligned}$$

From (2.17), (2.23) and (2.16), we have

(2.24)

$$\begin{aligned} \bar{V}^i &= V^i - F^i_{hj}(x,\xi)V^h\xi^j - F^i_{hj}(x,\xi)V^h\eta^j + F^i_{hj}(x,-\eta)V^h\xi^j + F^i_{hj}(x,-\eta)V^h\eta^j \\ &+ \frac{\partial F^i_{hj}}{\partial y^k}(x,-\eta)(\eta^k-\xi^k)V^h\xi^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(\eta^k-\xi^k)V^h(-\eta^j) \\ &+ \frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\eta^kV^h\xi^j - \frac{\partial F^i_{hk}}{\partial x^j}(x,\xi)\xi^jV^h\eta^k + F^i_{mk}(x,\xi)F^m_{hj}(x,\xi)V^h\xi^j\eta^k - F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)V^h\xi^j\eta^k \\ &- F^i_{mj}(x,-\eta)F^m_{hk}(x,\xi)V^h(\xi^k+\eta^k)(\xi^j+\eta^j) + F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)V^h(\xi^j+\eta^j)(\xi^k+\eta^k) + \cdots \\ &= V^i - F^i_{hj}(x,\xi)V^h\xi^j - F^i_{hj}(x,\xi)V^h\eta^j + F^i_{hj}(x,-\eta)V^h\xi^j + F^i_{hj}(x,-\eta)V^h\eta^j \\ &+ \frac{\partial F^i_{hj}}{\partial y^k}(x,-\eta)(\eta^k-\xi^k)V^h\xi^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(\eta^k-\xi^k)V^h(-\eta^j) \\ &+ \frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta)\eta^kV^h\xi^j - \frac{\partial F^i_{hk}}{\partial x^j}(x,\xi)\xi^jV^h\eta^k + F^i_{mk}(x,\xi)F^m_{hj}(x,\xi)V^h\xi^j\eta^k - F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta)V^h\xi^j\eta^k \\ &- F^i_{mj}(x,-\eta)(F^m_{hk}(x,\xi) - F^m_{hk}(x,-\eta))V^h(\xi^k+\eta^k)(\xi^j+\eta^j) + \cdots. \end{aligned}$$

Further, if we use $F^m_{hk}(x,\xi) = F^m_{hk}(x,-\eta) + \dfrac{\partial F^m_{hk}}{\partial y^a}(x,-\eta)(\xi^a+\eta^a)+\cdots$, then the last term of (2.24) disappears. Then we have

Proposition 2.3 *Let M be an n-dimensional differentiable manifold with a Finsler connection $F\Gamma = (N^i_j(x,y), F^i_{jk}(x,y), C^i_{jk}(x,y))$ satisfying $T^i_{jk}(x,y)=0$. For an infinitesimal parallelogram defined by (II) and an initial vector $V=(V^i)$, the parallel vector $\bar V = (\bar V^i)$ is obtained from a linear parallel displacement along c. Then $\bar V$ and V have the following relation*

(2.25)

$$\bar V^i - V^i = [(F^i_{hj}(x,-\eta) - F^i_{hj}(x,\xi))(\xi^j+\eta^j) + (\frac{\partial F^i_{hj}}{\partial y^k}(x,-\eta)\xi^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(-\eta^j))(\eta^k-\xi^k)$$
$$+(\frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta) - \frac{\partial F^i_{hk}}{\partial x^j}(x,\xi) - F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta) + F^i_{mk}(x,\xi)F^m_{hj}(x,\xi))\xi^j\eta^k]V^h + \cdots .$$

Now, we put quantities A and B as follows

$$\begin{aligned}A :=\ & (F^i_{hj}(x,\xi) - F^i_{hj}(x,\eta))(\xi^j+\eta^j) + (\frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)\eta^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\eta)\xi^j)(\eta^k-\xi^k)\\ &+(\frac{\partial F^i_{hj}}{\partial x^k}(x,\xi) - \frac{\partial F^i_{hk}}{\partial x^j}(x,\eta) - F^i_{mj}(x,\xi)F^m_{hk}(x,\xi) + F^i_{mk}(x,\eta)F^m_{hj}(x,\eta))\eta^j\xi^k\end{aligned} \tag{2.26}$$

(2.27)

$$B := (F^i_{hj}(x,-\eta) - F^i_{hj}(x,\xi))(\xi^j+\eta^j) + (\frac{\partial F^i_{hj}}{\partial y^k}(x,-\eta)\xi^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(-\eta^j))(\eta^k-\xi^k)$$
$$+(\frac{\partial F^i_{hj}}{\partial x^k}(x,-\eta) - \frac{\partial F^i_{hk}}{\partial x^j}(x,\xi) - F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta) + F^i_{mk}(x,\xi)F^m_{hj}(x,\xi))\xi^j\eta^k$$

We investigate a tensorial property of A and B in the next section.

3 Tensorial property of A and B

Let $\{U,(x^i,y^i)\}$ be a coordinate neighborhood on TM and $\{\bar U,(\bar x^i,\bar y^i)\}$ another one. We assume that

$$\bar x^i = \bar x^i(x) \ \ \text{(a coordinate transformation on } M),$$
$$\bar\xi^i = \xi^j\frac{\partial \bar x^i}{\partial x^j},\ \ \bar\eta^i = \eta^j\frac{\partial \bar x^i}{\partial x^j},$$

$\{U,(x^i,y^i)\} \ni (x^i,\xi^i),\ (x^i,\eta^i),\ (x^i+\xi^i,\eta^i),\ (x^i+\eta^i,\xi^i),\ (x^i+\xi^i+\eta^i,-\xi^i),\ (x^i+\eta^i,-\eta^i),\ (x^i,-\eta^i),$

$\{\bar U,(\bar x^i,\bar y^i)\} \ni (\bar x^i,\bar\xi^i),\ (\bar x^i,\bar\eta^i),\ (\bar x^i+\bar\xi^i,\bar\eta^i),\ (\bar x^i+\bar\eta^i,\bar\xi^i),\ (\bar x^i+\bar\xi^i+\bar\eta^i,-\bar\xi^i),\ (\bar x^i+\bar\eta^i,-\bar\eta^i),\ (\bar x^i,-\bar\eta^i).$

Then the following equations

$$\bar F^a_{bc}(\bar x,\bar y) = F^i_{jk}(x,y)\frac{\partial \bar x^a}{\partial x^i}\frac{\partial x^j}{\partial \bar x^b}\frac{\partial x^k}{\partial \bar x^c} + \frac{\partial \bar x^a}{\partial x^i}\frac{\partial^2 x^i}{\partial \bar x^b\partial \bar x^c}, \tag{3.1}$$

$$\frac{\partial \bar F^a_{bc}}{\partial \bar y^d}(\bar x,\bar y) = \frac{\partial F^i_{jk}}{\partial y^l}(x,y)\frac{\partial \bar x^a}{\partial x^i}\frac{\partial x^j}{\partial \bar x^b}\frac{\partial x^k}{\partial \bar x^c}\frac{\partial x^l}{\partial \bar x^d} \tag{3.2}$$

are satisfied, where $x=(x^i),\ y=(y^i),\ \bar{x}=(\bar{x}^i),\ \bar{y}=(\bar{y}^i)$. Therefore from (3.1)

$$\bar{F}^a_{bc}(\bar{x},\bar{\xi})-\bar{F}^a_{bc}(\bar{x},\bar{\eta})=(F^i_{jk}(x,\xi)-F^i_{jk}(x,\eta))\frac{\partial\bar{x}^a}{\partial x^i}\frac{\partial x^j}{\partial\bar{x}^b}\frac{\partial x^k}{\partial\bar{x}^c}\tag{3.3}$$

is satisfied. This means that the first part $(F^i_{hj}(x,\xi)-F^i_{hj}(x,\eta))(\xi^j+\eta^j)$ of A is a tensor field. Further from (3.2) we can see the fact the second part $(\frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)\eta^j+\frac{\partial F^i_{hj}}{\partial y^k}(x,\eta)\xi^j)(\eta^k-\xi^k)$ of A is also a tensor field.

On the other hand, in Finsler geometry, the third part of A

$$\frac{\partial F^i_{hj}}{\partial x^k}(x,\xi)-\frac{\partial F^i_{hk}}{\partial x^j}(x,\eta)-F^i_{mj}(x,\xi)F^m_{hk}(x,\xi)+F^i_{mk}(x,\eta)F^m_{hj}(x,\eta)\tag{3.4}$$

is not a tensor field. We, however, can calculate as follows

(3.5)

$$\begin{aligned}
&(\frac{\partial F^i_{hj}}{\partial x^k}(x,\xi)-\frac{\partial F^i_{hk}}{\partial x^j}(x,\eta)-F^i_{mj}(x,\xi)F^m_{hk}(x,\xi)+F^i_{mk}(x,\eta)F^m_{hj}(x,\eta))\eta^j\xi^k\\
&=(\frac{\delta F^i_{hj}}{\delta x^k}(x,\xi)+N^l_k(x,\xi)\frac{\partial F^i_{hj}}{\partial y^l}(x,\xi)-\frac{\delta F^i_{hk}}{\delta x^j}(x,\eta)-N^l_j(x,\eta)\frac{\partial F^i_{hk}}{\partial y^l}(x,\eta)\\
&\qquad -F^i_{mj}(x,\xi)F^m_{hk}(x,\xi)+F^i_{mk}(x,\eta)F^m_{hj}(x,\eta))\eta^j\xi^k\\
&=(\frac{\delta F^i_{hj}}{\delta x^k}(x,\xi)+(\xi^m F^l_{mk}(x,\xi)-D^l_k(x,\xi))\frac{\partial F^i_{hj}}{\partial y^l}(x,\xi)-\frac{\delta F^i_{hk}}{\delta x^j}(x,\eta)\\
&\qquad -(\xi^m F^l_{mj}(x,\eta)-D^l_j(x,\eta))\frac{\partial F^i_{hk}}{\partial y^l}(x,\eta)-F^i_{mj}(x,\xi)F^m_{hk}(x,\xi)+F^i_{mk}(x,\eta)F^m_{hj}(x,\eta))\eta^j\xi^k\\
&=(\frac{\delta F^i_{hj}}{\delta x^k}(x,\xi)-\frac{\delta F^i_{hk}}{\delta x^j}(x,\eta)-F^i_{mj}(x,\xi)F^m_{hk}(x,\xi)+F^i_{mk}(x,\eta)F^m_{hj}(x,\eta)\\
&\qquad\qquad -D^l_k(x,\xi)\frac{\partial F^i_{hj}}{\partial y^l}(x,\xi)+D^l_j(x,\eta)\frac{\partial F^i_{hk}}{\partial y^l}(x,\eta))\eta^j\xi^k\\
&\qquad\qquad +(F^l_{mk}(x,\xi)\frac{\partial F^i_{hj}}{\partial y^l}(x,\xi)-F^l_{mj}(x,\eta)\frac{\partial F^i_{hk}}{\partial y^l}(x,\eta))\xi^m\eta^j\xi^k.
\end{aligned}$$

In our calculations, we can ignore 3rd order terms with respect to ξ,η. Then the quantity

$$\begin{aligned}&\frac{\delta F^i_{hj}}{\delta x^k}(x,\xi)-\frac{\delta F^i_{hk}}{\delta x^j}(x,\eta)-F^i_{mj}(x,\xi)F^m_{hk}(x,\xi)+F^i_{mk}(x,\eta)F^m_{hj}(x,\eta)\\
&-D^l_k(x,\xi)\frac{\partial F^i_{hj}}{\partial y^l}(x,\xi)+D^l_j(x,\eta)\frac{\partial F^i_{hk}}{\partial y^l}(x,\eta)\end{aligned}\tag{3.6}$$

is a tenor field in Finsler geometry. So we have

(3.7)

$$\bar{V}_r^i - V_r^i = [(F^i_{hj}(x,\xi) - F^i_{hj}(x,\eta))(\xi^j + \eta^j) + (\frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)\eta^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\eta)\xi^j)(\eta^k - \xi^k)$$
$$+ (\frac{\delta F^i_{hj}}{\delta x^k}(x,\xi) - \frac{\delta F^i_{hk}}{\delta x^j}(x,\eta) - F^i_{mj}(x,\xi)F^m_{hk}(x,\xi) + F^i_{mk}(x,\eta)F^m_{hj}(x,\eta)$$
$$- D^l_k(x,\xi)\frac{\partial F^i_{hj}}{\partial y^l}(x,\xi) + D^l_j(x,\eta)\frac{\partial F^i_{hk}}{\partial y^l}(x,\eta))\eta^j\xi^k]V^h + \cdots .$$

By the similar calculation for B, we have

(3.8)

$$\bar{V}^i - V^i = [(F^i_{hj}(x,-\eta) - F^i_{hj}(x,\xi))(\xi^j + \eta^j) + (\frac{\partial F^i_{hj}}{\partial y^k}(x,-\eta)\xi^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(-\eta^j))(\eta^k - \xi^k)$$
$$+ (\frac{\delta F^i_{hj}}{\delta x^k}(x,-\eta) - \frac{\delta F^i_{hk}}{\delta x^j}(x,\xi) - F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta) + F^i_{mk}(x,\xi)F^m_{hj}(x,\xi)$$
$$- D^l_k(x,-\eta)\frac{\partial F^i_{hj}}{\partial y^l}(x,-\eta) + D^l_j(x,\xi)\frac{\partial F^i_{hk}}{\partial y^l}(x,\xi))\xi^j\eta^k]V^h + \cdots .$$

Therefore we have

Proposition 3.1 *Let M be an n-dimensional differentiable manifold with a Finsler connection $F\Gamma = (N^i_j(x,y), F^i_{jk}(x,y), C^i_{jk}(x,y))$ satisfying $T^i_{jk}(x,y) = 0$. First, for an infinitesimal parallelogram defined by (I) and an initial vector $V = (V^i)$, we have vectors $V_r = (V^i_r), \bar{V}_r = (\bar{V}^i_r)$ and the difference $\bar{V}_r - V_r$ satisfies* (3.7). *Next, for an infinitesimal parallelogram defined by (II) and an initial vector $V = (V^i)$, the parallel vector $\bar{V} = (\bar{V}^i)$ is obtained and the differences $\bar{V} - V$ satisfies* (3.8).

Finally, when the deflection tensor field $D^i_j = y^m F^i_{mj} - N^i_j$ vanishes, we have the following theorem

Theorem 3.1 *Let M be an n-dimensional differentiable manifold with a Finsler connection $F\Gamma = (N^i_j(x,y), F^i_{jk}(x,y), C^i_{jk}(x,y))$ satisfying $T^i_{jk}(x,y) = 0,\ D^i_j(x,y) = 0$. First, for an infinitesimal parallelogram defined by (I) and an initial vector $V = (V^i)$, we have vectors $V_r = (V^i_r), \bar{V}_r = (\bar{V}^i_r)$ and the difference $\bar{V}_r - V_r$ satisfies* (3.9). *Next, for an infinitesimal parallelogram defined by (II) and an initial vector $V = (V^i)$, the parallel vector $\bar{V} = (\bar{V}^i)$ is obtained and the differences $\bar{V} - V$ satisfies* (3.10).

(3.9)

$$\bar{V}_r^i - V_r^i = [(F^i_{hj}(x,\xi) - F^i_{hj}(x,\eta))(\xi^j + \eta^j) + (\frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)\eta^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\eta)\xi^j)(\eta^k - \xi^k)$$
$$+ (\frac{\delta F^i_{hj}}{\delta x^k}(x,\xi) - \frac{\delta F^i_{hk}}{\delta x^j}(x,\eta) - F^i_{mj}(x,\xi)F^m_{hk}(x,\xi) + F^i_{mk}(x,\eta)F^m_{hj}(x,\eta))\eta^j\xi^k]V^h + \cdots ,$$

(3.10)

$$\bar{V}^i - V^i = [(F^i_{hj}(x,-\eta) - F^i_{hj}(x,\xi))(\xi^j + \eta^j) + (\frac{\partial F^i_{hj}}{\partial y^k}(x,-\eta)\xi^j + \frac{\partial F^i_{hj}}{\partial y^k}(x,\xi)(-\eta^j))(\eta^k - \xi^k)$$
$$+ (\frac{\delta F^i_{hj}}{\delta x^k}(x,-\eta) - \frac{\delta F^i_{hk}}{\delta x^j}(x,\xi) - F^i_{mj}(x,-\eta)F^m_{hk}(x,-\eta) + F^i_{mk}(x,\xi)F^m_{hj}(x,\xi))\xi^j\eta^k]V^h + \cdots .$$

Remark 3.1 *Under the same conditions in Theorem 3.1, if we investigate the difference* $\bar{V}_r^i - V_r^i$ *obtained from the traditional way of a parallel displacement defined by* (1.6)*, then we have*

$$\bar{V}_r^i - V_r^i = R^i_{jk}(x, V)\xi^k \eta^j, \tag{3.11}$$

where $R^i_{jk}(x, y)$ *is one of torsion tensor fields, namely,* $R^i_{jk}(x, y) = \frac{\delta N^i_j}{\delta x^k}(x, y) - \frac{\delta N^i_k}{\delta x^j}(x, y)$.

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